Quality Ladders, Competition and Endogenous Growth[1]

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Abstract: We examine a theory of competitive innovation in which new ideas are introduced only when diminishing returns to the use of existing ideas sets in. After an idea is introduced, the knowledge capital associated with that idea expands, and its value falls. Once the value falls far enough, it becomes profitable to introduce a new idea. The resulting theory is consistent with fixed costs of innovation and it accounts for the same facts as existing theories of endogenous growth. However, there is evidence that innovation frequently takes place even absent monopoly power and that it is driven by diminishing returns on existing ideas – two facts that the existing theory cannot account for.

[1] We are indebted to years of discussion with Chad Jones, Larry Jones, Robert E. Lucas, Joseph Ostroy, Paul Romer, and James A. Schmitz about these issues. Pete Klenow, Rafael Repullo and Nancy Stokey also made many useful comments on this paper. We are especially grateful to Alexandre Monge-Naranjo for a masterful discussion in New York, which we have used extensively. Thanks to the National Science Foundation, Grant No. SES-03-14713, and the Spanish MES, Grants No. SEJ 2005-08783-C04-01 and ECO2008-06395-C05-01, for financial support.

1. Introduction

The standard view of innovation is that progress takes place along a quality ladder – driven by the short-term monopoly power innovators acquire at each step, as short-term monopoly power is essential to compensate for the fixed cost of innovating. For nearly sixty years, since it was first advanced in Schumpeter [1942], this theory has been the primary tool accounting for the dependence of technological progress on economic fundamentals such as patience and cost. This picture of innovation has been used as an explanation for economic growth in the models of Romer [1990], Grossman and Helpman [1991], and Aghion and Howitt [1992], among other.

Our own and other researchers examination of innovation[2] suggests that a different story is worthy of consideration. No doubt, each innovation opens the door to moving on to a new rung of the quality ladder. However, when innovation is costly, it is socially, and most often privately, best to exploit the adoption and imitation opportunities it makes available on the current rung before moving on to the next. Only when these are exhausted does it becomes socially and privately optimal to introduce a new innovation. In such a process fixed costs and monopoly power play no essential role.

There are abundant examples to illustrate the different predictions of the two theories, the most obvious being that in the standard theory absent monopoly power innovation is impossible, while – as occurs in practice – it is possible in the other case. More relevant to the quality ladder framework, the standard view predicts that, for example, after the basic Bessemer process for steelmaking was invented[3], innovators' attention and resources would be immediately devoted to the basic oxygen steelmaking process, which instead came around only a century later. Our theory predicts, instead, that they would, as they did, continue to spend their time and energy improving the Bessemer process, expanding its usage in the production of steel and, more generally, exploit its adoption in new profitable venues. Only after the Bessemer process was both greatly improved and widespread, and the gains to its further improvement and expansion driven down, would potential innovators move on to work on a completely different

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[2] See, for example, our recent book: Boldrin and Levine [2008b] as well as, among the very many others, Payson Usher [1929], Burn [1961], Mansfield [1986], Bessen and Maskin [1999], Herrera and Schroth [2003], McClellan and Dorn [2006], Braguinsky et al. [2007].

[3] In the late 1850s, see http://en.wikipedia.org/wiki/Steel and references therein.

method for steelmaking. The latter is an accurate description of the historical facts pertaining to the invention of the two methods, as well as of radio and television, steam power engines, computer processors, aircraft engines and so on – in fact, the history of R&D shows it is the rule and not the exception. After the successful introduction of a new product, potential innovators do not turn immediately to inventing something else that may replace it, but rather to adopting, imitating and possibly improving the recent invention. After inventing the light bulb, Thomas Edison turned primarily to investing in and promoting electrical power and selling light bulbs – not to inventing the fluorescent light or the LED. His would-be competitors did likewise. As every movie producer can tell you, the time to release your great new blockbuster adventure movie is not a few days after your rival has done the same.

The theory of competitive innovation is not new. Aside from the classical theorists – Schumpeter [1912], von Neumann [1937], Plant [1934] and Stigler [1956], among others – recent important contributions include the unpublished works of Funk [1996] and Quah [2002], the paper by Hellwig and Irmen [2001], and our own work, Boldrin and Levine [1997, 2002 and 2008a]. Examples of papers in which competitive innovation is applied to substantive questions other than endogenous growth are, Andolfatto and McDonald [1998], Boldrin and Levine [2001], Funk [2008], Hopenhayn [2006], Jovanovic and McDonald [1994], Prescott and McGrattan [2008]. During the last twenty years, many other authors have contributed to this body of literature by investigating various of its aspects, such as the connections between constant returns to scale and economic growth (Jones and Manuelli [1990], Rebelo [1991]), the link between the degree of appropriation and competition (Makowski and Ostroy [1995, 2001]), and the role of indivisibilities in economic development (Acemoglu and Zilibotti [1997]).

The goal of our earlier work was largely to demonstrate the possibility of competitive innovation by clarifying the role of limited capacity in determining competitive rents. Our goal in this paper is to advance the theory to empirical relevance by embodying it in a benchmark growth model that – by capturing the idea that advances are not desirable until existing opportunities have been exploited – can help account for the facts of innovation.

2. The Grossman-Helpman Model

Because the predictions of standard models of endogenous growth are well know, we use them as a yardstick against which to evaluate those of the model we propose. There are a variety of models of quality ladders with fixed costs, increasing returns, and external effects, most notably those of Romer [1990], Grossman and Helpman [1991], and Aghion and Howitt [1992]. We take as our starting point the one by Grossman and Helpman [1991] because it leads to a simple closed form solution allowing for a straightforward welfare analysis. Here we summarize their results, employing their notation throughout.

Goods come in different qualities.[4] Denote by d_j the consumption of goods of quality j, let ρ be the subjective interest rate, and $\lambda > 1$ the (constant) increase in quality as we move one step up the ladder. We let

$$c_t = \sum_j \lambda^j d_{jt}$$

denote aggregate consumption. Utility of the representative consumer is

$$U = \int_0^\infty e^{-\rho t} \log[\, c_t \,] dt \,.$$

One unit of output of each quality requires one unit of labor to obtain. The first firm to reach step j on the quality ladder is awarded a legal monopoly over that technology. This monopoly lasts only until there is a new innovation and technology $j+1$ is introduced, at which time all firms have access to technology j. Taking labor to be the numeraire, the implication is that the price of output of technology $j+1$ relative to that of technology j is given by the limit-pricing formula $p = \lambda$.

The intensity of R&D for a firm is denoted by $\tilde{\iota}$, and the probability of successfully achieving the next step during a period of length dt is $\tilde{\iota}dt$ at a cost of $\tilde{\iota} a_I dt$.

Let E denote steady state consumer spending. Since the wage rate is numeraire and price is λ the monopolist gets a margin of $\lambda - 1$ on each unit sold. His share of E is therefore his margin divided by the price $(\lambda - 1)/\lambda = 1 - 1/\lambda$. Since the cost of getting the monopoly is a_I, the rate of return is $(1 - 1/\lambda)E/a_I$. However, there is a

[4] In the original Grossman-Helpman paper there was a continuum of identical sectors indexed by ω. Since this plays little role in the analysis, and for notational simplicity, we omit it here.

chance ι of losing the monopoly, reducing the rate of return by this amount. Equating this net rate of return to the subjective interest rate determines research intensity

$$\frac{(1-1/\lambda)E}{a_I} - \iota = \rho\,.$$

There is a single unit of labor,[5] the demand for which comes from the $a_I\iota$ units used in R&D and the E/λ units used to produce output, yielding the constraint

$$a_I\iota + E/\lambda = 1\,.$$

Notice that the same labor is used for R&D as is used to produce output. This captures the sensible – and in this model crucial – idea that there is an increasing marginal cost of R&D. That the increasing cost is due to resources being sucked out of the output sector is analytically convenient. It implies that the cost of R&D, measured in units of output, is proportional to the current rung on the quality ladder, making possible steady state analysis.

These two equations can be solved for the steady state research intensity

$$\iota = \frac{(1-1/\lambda)}{a_I} - \frac{\rho}{\lambda}.$$

By contrast the socially optimal research intensity is derived by calculating steady state utility to be $[\log E - \log\lambda + (\iota/\rho)\log\lambda]/\rho$. Since the optimal plan in a steady state maximizes the steady state utility subject to the resource constraint, simple algebra gives the optimum

$$\iota^* = \frac{1}{a_I} - \frac{\rho}{\log\lambda}.$$

3. Climbing the Ladder under Competition

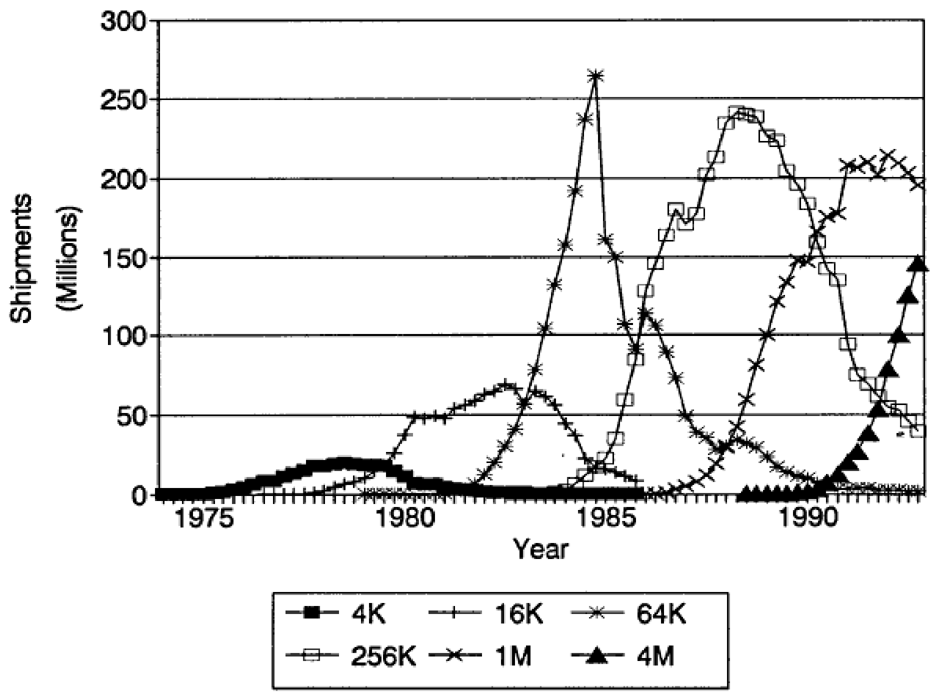
Innovation is driven by the rush of firms to get ahead of one another, suggesting that competition is the driving force. In the story just summarized, the reward from competing is some temporary monopoly power. We want to tell an alternative story in which the incentives for innovating come about because of diminishing returns to making use of previous inventions, and in which the reward from competing is a higher return on capital. Because inventing a new good costs more than reproducing an existing one, it

[5] We have simplified Grossman and Helpman notation by normalizing the stock of labor to one.

becomes profitable to invent the new good only when the available quantity of the old good is large enough to make its price low, relative to that of the new one.

Before presenting a model of endogenous innovation due to diminishing returns, it is useful to look at a typical example of how real quality ladders work, borrowed from Irwin and Klenow [1994]. The good in question is the DRAM memory chip. The

Figure 1: DRAM Vintages



different qualities correspond to the capacity of a single chip. Figure 1, showing shipments of different quality chips, is reproduced from that paper. The key fact is that production of a particular quality does not jump up instantaneously but ramps up gradually, and that a new quality is introduced when the stock of the old one is fairly large. Further, the old vintage is phased out gradually as the new one is introduced. Their price data shows that the price of each vintage of chip falls roughly exponentially over the product cycle – meaning that the incentive to introduce the next generation chip keeps increasing. This vividly portrays our story. Evidence suggests this is the usual pattern in most industries.[6] The question this evidence poses, and the intuition upon which our model is built is: why introduce a new product if the old one is still doing so well?

4. Innovation with Knowledge Capital

We adopt the same demand structure as Grossman and Helpman, hence quality adjusted consumption is

$$c_t = \sum_j \lambda^j d_{jt}$$

and the preferences for the representative consumer are

$$U = \int_0^\infty e^{-\rho t} \log[c_t] dt .$$

Unlike Grossman and Helpman, we assume that consumption is produced both from labor and the existing stock of specialized productive capacity. For simplicity we identify "productive capacity" with knowledge and assume that different rungs on the quality ladder correspond to different qualities of capital and knowledge used to produce that particular consumption good. We denote by k_j the combined stock of capital and embedded knowledge that goes into producing quality j output. By explicitly modeling the stock of knowledge, we can distinguish between investment on a given rung – spreading and adopting knowledge of a given type through teaching, learning, imitation, and copying – and investment that moves between rungs – innovation or the creation of new knowledge. We refer to k_j as *quality* j *knowledge capital* or, simply, *knowledge.* In practice this can have many forms – it can be in the form of human knowledge or human capital, but it can also be embodied in physical forms such as books, or factories and machines with the appropriate design. In our theory it plays the crucial role of keeping track of progress on a particular rung of the quality ladder.

Knowledge has two uses: it can be used either to generate more knowledge or to produce consumption. More knowledge is produced either by increasing the stock of the same quality of knowledge capital or by creating a higher quality. If quality j knowledge is used to produce more knowledge of the same quality, it does so at a fixed rate $b > \rho$

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[6] See, for example, Hannan and McDowell [1987], Manuelli and Seshadri [2003], Rose and Joskow [1990], Sarkar [1998]. This is closely related to the S-shaped diffusion curve for new products that – since the early work of Rogers [1962] – is widely accepted as a stylized fact; see Jovanovic and Lach [1989], for a recent economic model.

per unit of input. In other words the production function for existing knowledge is linear in the knowledge capital used as input. This can be regarded as *capital widening* or accumulation and/or competitive imitation.[7]

More knowledge can also be created through innovation – that is, the production of a higher quality of knowledge capital from an existing quality. This can be regarded as *capital deepening*. Specifically, a unit of knowledge capital of quality $j+1$ can be produced from $a > \lambda$ units of quality j. In the strict interpretation of k_j as a stock of knowledge about how to produce, capital widening corresponds to the spread of existing knowledge, and capital deepening to the creation of new knowledge from old.[8]

Alternatively, knowledge capital of quality j can be employed in the production of quality j consumption on a one-to-one basis. The production of consumption also requires labor – leading here to diminishing returns for each quality of knowledge capital. Note that it is the diminishing returns on each rung of the ladder that is crucial to our story: the specific details of diminishing returns arising from a labor constraint are used for analytical convenience. Specifically, each unit of quality j knowledge capital employed in the consumption sector requires also a unit of labor to produce a unit flow of quality j consumption. Our key assumption is that measured in units of current consumption the creation of new knowledge is costlier than spreading knowledge already in existence, that is $b > \lambda / a.$ This implies that as long as it is not needed for expanding consumption – that is, until all labor is employed with the most advanced quality of knowledge capital – it is not socially efficient to introduce a higher quality of knowledge in the production of consumption. We also normalize the fixed labor supply to one.

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[7] As before, the assumption of linearity is purely for algebraic convenience: any sufficiently productive concave function would also do and, in fact, enrich the model from an applied perspective.

[8] Notice that new knowledge capital loses the capability of the old knowledge capital that was converted. This may be true for the physical replacement of machines with newer models, but is not usually the case for human capital – a pilot may still be capable of flying a Cessna after learning to fly a Boeing 777. However, the key constraint is that at any moment in time you must be acting as one of the two kinds of pilot, but not as both. If we introduced a technology for converting quality $j+1$ knowledge capital back to quality j at the same ratio as the forward conversion, then this would precisely capture knowledge that was not lost, but knowledge capital as a resource that could be deployed at only one level at a time. However, in the case we study, there is no reason to use the backward conversion technology, so the equilibrium would not change. Should there be a indivisibility in the deepening technology, the possibility of such a backward conversion would effectively eliminate the indivisibility: just produce enough of the new knowledge to overcome the indivisibility, then convert the excess back to the old knowledge. This may help explain why, in practice, indivisibilities do not seem to be an important practical problem for the bulk of innovation.

Let k_j^w, k_j^d and k_j^c denote, respectively, the investment of knowledge capital of quality j used in the production of (i) knowledge capital of quality j (widening), (ii) knowledge capital of quality $j+1$ (deepening), and, (iii) final consumption of quality j. At every point in time, the resource constraint is

$$k_{j,t} = k_{j,t}^w + k_{j,t}^d + k_{j,t}^c,$$

where the right hand side variables are all non negative and one or more of them may be chosen to be zero. As time is continuous, we omit the time subscript when not strictly needed.

Under our assumptions the motion over time of quality j stock of knowledge is given by

$$\dot{k}_j = bk_j^w + \frac{k_{j-1}^d}{a} - k_j^d.$$

Recall that $k_j^c = d_j$ units of the stock of knowledge must be allocated to the consumption sector. Note that, over a short period of time, there is no limit on the flow of quality j knowledge into quality $j+1$, that is k_j^d may be arbitrarily large, provided $\dot{k}_j$ is correspondingly negative, so we allow also discrete conversion $\Delta k_{j+1} = -\Delta k_j / a$.

The key technical fact is that this economy is an ordinary diminishing return economy with three sectors: widening, deepening and consumption. Competitive equilibrium can be described in the usual way as the combination of consumer optimization and profit maximization. The first and second welfare theorems hold, so efficient allocations can be decentralized as a competitive equilibrium and vice versa.[9]

5. Competitive Equilibrium with Knowledge Capital

Our primary goal is to characterize the competitive equilibrium of the model – or equivalently the solution to the planner problem. We will examine the robustness of the equilibrium to the main assumptions later. For the sake of the reader, we summarize here our three parametric assumptions and the purposes they serve: (i)

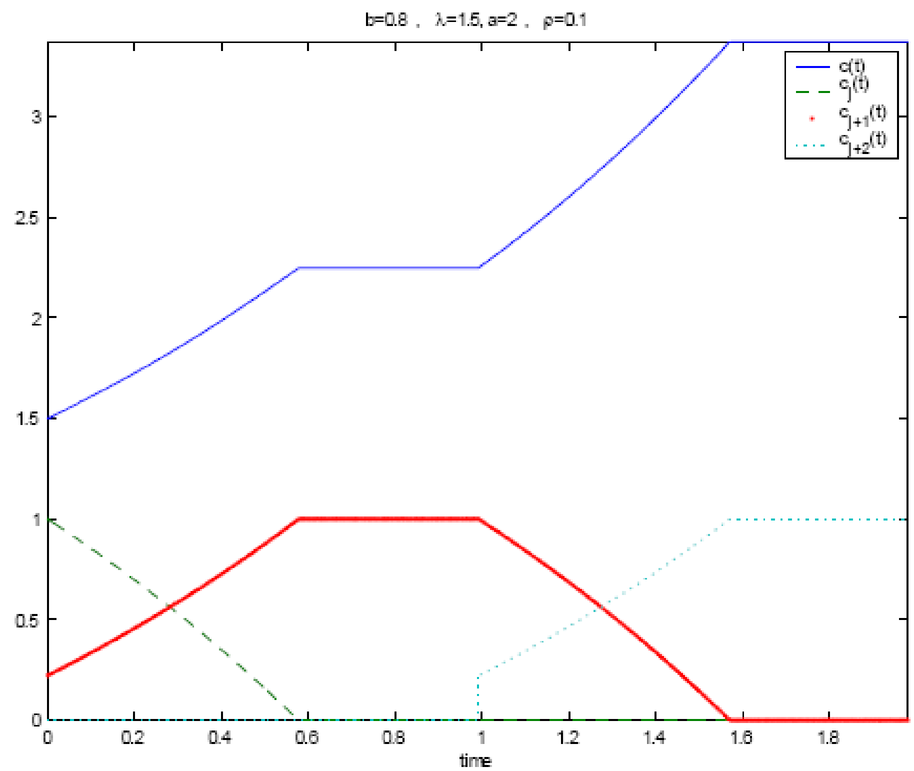
[9] The welfare theorems are proven for the discrete time version of this model by Boldrin and Levine [2002]. In that paper we assumed that one unit of capital of type j and λ^{-j} units of labor were required to produce a unit of quality j consumption, and that the capital is used up in the process of producing consumption. This simply changes the units in which knowledge capital is measured. The assumptions here are chosen for compatibility with Grossman and Helpman [1991].

$a > \lambda$, makes deepening costly; (ii) $b > \rho$, makes widening profitable; (iii) $b > \lambda / a$, makes deepening costlier than widening.

Under these assumptions we will show that, as the stock of knowledge capital grows, after a possible *initial unemployment phase* where there is too little knowledge to employ the entire stock of labor, the competitive equilibrium settles into a recurring cycle.

The cycle alternates between a *growth* phase (widening) in which consumption grows at the rate $b - \rho$ by upgrading the stock of knowledge used to produce it, and a *build-up* phase (deepening) in which consumption remains flat while a new quality of

Figure 2: Aggregate consumption and consumption of different vintages



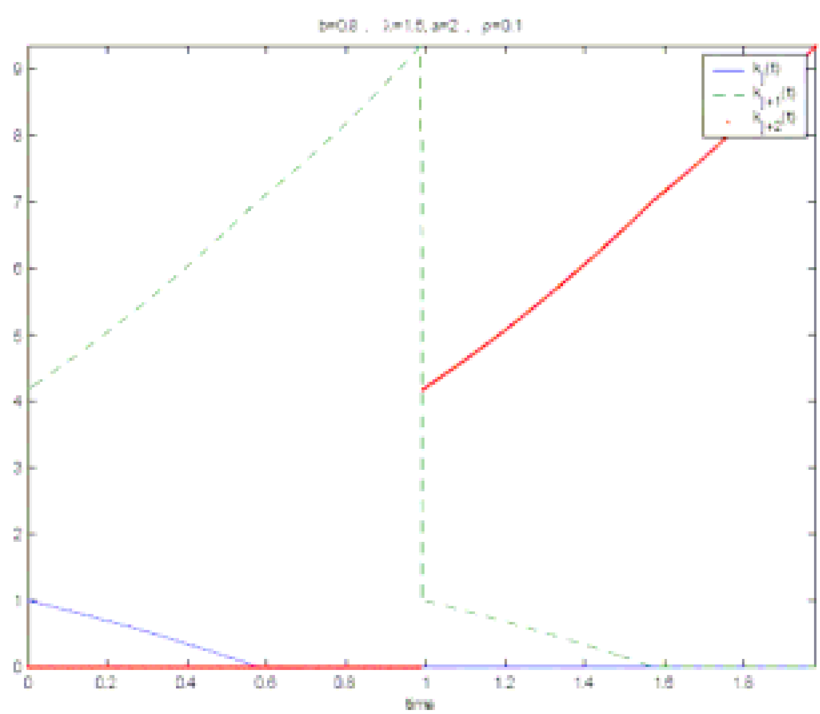
knowledge capital is accumulated. The growth phase ends when it is no longer possible to increase consumption without innovation - that is when all labor is applied to the most advanced type of knowledge capital, say quality j. This behavior of consumption is illustrated (for the case $b = 0.8, \lambda = 1.5, a = 2.0, \rho = 0.1$) in Figure 2[10], where the blue

[10] Thanks to Alexander Monge Naranjo for producing Figures 2 and 3 while discussing this paper at the Economic Development Conference, NYU-Stern, March 2009.

line describes aggregate consumption and the green dashed, red and blue dotted lines portray, respectively, the time paths of c_j, c_{j+1} and c_{j+2}.

To understand how innovation takes place – that is, knowledge capital of quality $j+1$ is first created, then progressively accumulated and applied to the production of consumption by replacing knowledge capital of quality j – the key idea is that the relative value of quality j and $j+1$ knowledge capital simultaneously used in producing consumption is λ, while the instantaneous price ratio, when the innovation first occurs, must be $a > \lambda$. This has an important implication. Because $a > \lambda$, at the first moment that the switchover from quality $j-1$ to quality j knowledge capital is complete, using knowledge capital of quality $j+1$ to replace knowledge capital of quality j in the production of consumption would yield negative profits. A positive amount of time – the "build up phase" – must elapse between the exhaustion of the $j-1$ technology and the use of knowledge capital of quality $j+1$ to produce consumption. During this build up phase the new type of knowledge is accumulated and its price drops until it becomes cheap enough to use it in the production of consumption. This begins the

Figure 3: Evolution of the stock of capital of different vintages



next growth phase, which must therefore begin a discrete amount of time after the end of the previous one.[11]

This behavior of knowledge capital is illustrated (again for the case $b = 0.8, \lambda = 1.5, a = 2.0, \rho = 0.1$) in Figure 3, where the blue line, the green dashed and red lines portray, respectively, the time paths of k_j, k_{j+1} and k_{j+2}. The reader should note especially the jumps in the level of old and new capital stock at the time of innovation, which are carefully discussed later and constitute a key prediction of our model.

A useful measure of the rate of technological progress is research intensity. In this setting it is measured by the inverse of the combined length τ^* of the two phases. We will compute it to be

$$i^* = 1 / \tau^* = \frac{b - \rho}{\log a},$$

so that it is increasing with b the rate of capital widening, and decreasing with the subjective interest rate ρ and the cost of knowledge creation a.

The Pricing of Knowledge Capital

In the next three propositions we wish to characterize the competitive equilibrium – or equivalently the solution to the planner problem. The Lagrange multipliers supporting the optimal quantities are of course the competitive equilibrium prices; it turns out to be more convenient to solve the model in terms of prices instead of quantities. These prices satisfy the two key competitive properties: they support household consumption and yield zero profits. We exploit these two properties, and the model's linearity, to solve it analytically.

The key to understanding the competitive equilibrium is the pricing of knowledge capital. It is convenient not to take labor as numeraire, but rather use current utility to that purpose. This implies that the current price of consumption is marginal utility; in our logarithmic case equal to $1/c_t$. Define q_{jt} to be the time t price of quality j knowledge capital. We first examine the value of knowledge in the production of more knowledge.

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[11] We are grateful to V.V. Chari for forcing us to clarify the intuitive reason why the build up phase must have a strictly positive length.

One use of knowledge is for deepening – that is to innovate by creating higher quality knowledge capital. *Zero profits on deepening* implies that $q_{j+1,t} - aq_{jt} \leq 0$, that is: $q_{j+1,t} / q_{jt} = a$ at the time of innovation. Knowledge can also be used for widening – that is to create more knowledge of the same quality. In this case the physical rate of return is the growth rate b, so we must have this return plus capital gains equal to the subjective interest rate, that is, $b + \dot{q}_{jt} / q_{jt} = \rho$, or equivalently for *zero profits on widening* $\dot{q}_{jt} / q_{jt} = -(b - \rho) < 0$.[12]

That the price of knowledge is necessarily falling over time is significant. Consider, for example, the fact that the first mover must incur greater costs than subsequent competitors. This is true here, since, to produce quality $j+1$ from quality j, the first innovator must pay more than subsequent imitators for knowledge as an input. However: by virtue of being first, he can also sell his freshly created knowledge for a higher price than his imitators, who must sell at a later date when the new knowledge is worth less. In other words, even in this model of perfect competition, there is a first mover advantage, because the competitive price of knowledge capital falls after it is first created. It is this first mover advantage that motivates innovators to act while knowing they will soon be imitated.

Consumption Value of Knowledge Capital: The Initial Unemployment Phase

Next we turn to the value and price of knowledge used to produce consumption. When the economy begins, the initial condition may be such that there is insufficient knowledge capital to employ all the labor. We first examine the value of knowledge used to produce consumption during this phase. Recall that, when they are both used in producing consumption, the relative value of quality j and $j+1$ knowledge is λ, while at the time the second is first created their price ratio must be $a > \lambda$. This implies that, when there is no full employment, only the lowest quality of knowledge is used to produce consumption. Subsequently we will establish that when there is full employment, no more than two qualities of knowledge capital are used to produce consumption.

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[12] Strictly speaking, these conditions need only hold with inequality if knowledge capital is not being used to produce more knowledge capital. For example, if there is no capital of quality J or higher, then we can have $q_{j+1,t} < aq_{jt}$ for $j+1 \geq J$. However, there exist equivalent equilibria in which profits are zero. See Boldrin and Levine [2002] for the technical details.

What is the value of knowledge capital used to produce consumption when there is unemployment? We can determine this from the fact that consumption and the knowledge capital producing it are perfect substitutes: both can and are produced from the current stock of knowledge.[13]

Proposition 1: *Depending on initial conditions, there may be an* initial unemployment phase *during which the single lowest quality of knowledge,* j*, is used to produce consumption. The price of knowledge capital is* $q_{jt} = v_{jt} = \lambda^j / bc_t$*. Consumption grows at the constant rate* $\dot{c}_t / c_t = b - \rho$*. The unemployment phase ends when* $c_t = \lambda^j$*.*

Proof: Consider how we might use a small amount ε of quality j knowledge over a short period of time τ to produce either consumption or more knowledge of quality j. If consumption is to be produced, ε units of quality j knowledge yield $\lambda^j \varepsilon\tau$ units of consumption. If quality j knowledge is to be produced the yield is $b\varepsilon\tau$ new units. Hence one unit of quality j knowledge is a perfect substitute for λ^j / b units of consumption.

Since the marginal social value of c_t units of quality adjusted consumption is $1/c_t$, we can conclude that, when there is unemployment, the marginal social value in producing consumption of a unit of quality j knowledge is $v_{jt} \equiv \lambda^j / bc_t$. Its price cannot be less than this: we must have $q_{jt} \geq v_{jt}$, with equality if knowledge of that quality is actually used to produce consumption. As claimed, if quality j' knowledge capital is used to produce consumption, then no higher quality can be used. That is, if quality $j > j'$ was used to produce consumption, then we would have $q_{jt} / q_{j't} = v_{jt} / v_{j't} = \lambda^{j-j'}$. But, because knowledge capital of quality j can be readily obtained from that of quality j' by applying the innovation technology for $j - j'$ times, the latter equality would contradict the condition for zero profit on deepening, $q_{jt} / q_{j't} = a^{j-j'}$.

We have shown that during the *initial unemployment phase* a single quality of knowledge, j, is used to produce consumption, the lowest available quality. The price of this knowledge capital is $q_{jt} = v_{jt} = \lambda^j / bc_t$. Since the condition for no profit on widening is $\dot{q}_{jt} / q_{jt} = -(b - \rho)$, it must be that consumption grows at the constant rate $\dot{c}_t / c_t = b - \rho$. Eventually, the stock of knowledge of quality j grows sufficiently large that a quantity of consumption $c_t = \lambda^j$ can be produced, meaning that we have reached

[13] By assumption the labor constraint is not binding during the unemployment phase.

full employment and it is no longer possible to increase output by employing more quality j knowledge.

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Consumption Value of Knowledge Capital: Full Employment

We next examine the value of knowledge capital used to produce consumption when there is full employment. Suppose that quality $j' < j$ knowledge is used to produce consumption. In this case, consumption may be increased by replacing some of the inferior quality j' knowledge capital with additional units of quality j knowledge, which has, therefore, value in producing consumption. Notice that, instead, the lowest quality knowledge used in producing consumption has no marginal value there: additional units cannot increase consumption at all.

We follow the same analysis as in the initial unemployment phase: we determine a price ratio between consumption and quality j knowledge capital by examining how much of each we can produce from a given small amount ε of quality j knowledge over a short period of time τ. The key difference is that with full employment when we move quality j knowledge into the consumption sector, we must displace some lower quality j' knowledge in order to free up the labor needed to work with the newly added quality j knowledge.[14] This frees the displaced j' knowledge, which, of course, has social value.

Proposition 2: *During full employment no more than two qualities of knowledge capital are actually used to produce consumption, and these must be consecutive qualities. If* j' *is used to produce consumption*

$$q_{jt} \geq v_{jt}^{j'} \equiv \frac{\lambda^j - \lambda^{j'}}{b(1 - 1/a^{j-j'})} \frac{1}{c_t},$$

with equality if j *is also used to produce consumption. If two qualities are both used in positive amounts in producing consumption the latter grows at the rate* $b - \rho$.

Proof: In Appendix 1.

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[14] This is often labeled "creative destruction": a more productive technology replaces a less productive one through the scrapping of the latter and the shifting of workers from old plants to new ones. In practice, this often involves replacing firms using the old quality capital with firms that have adopted the new one.

The Growth Cycle

We are now in a position to analyze what happens at the end of the initial unemployment phase when employment is full. Because, after this initial phase, the optimal program implies there is always full employment, the forthcoming analysis applies to all subsequent periods. That is, possibly after an initial unemployment phase, the economy cycles between growth phases - during which consumption grows at the constant rate $b-\rho$ - and build-up phases - where consumption remains constant. After each complete cycle, it repeats at the next level of the quality ladder.

Proposition 3: *Consumption remains constant during a build-up phase that lasts for*

$$\tau^b = \frac{\log a - \log \lambda}{b-\rho},$$

followed by a growth phase during which consumption grows at the rate $b-\rho$ *lasting*

$$\tau^g = \frac{\log \lambda}{b-\rho}$$

The total length of a cycle is

$$\tau^* = \frac{\log a}{b-\rho}$$

so that the resulting research intensity is

$$1/\tau^* = j^* = \frac{b-\rho}{\log a}$$

Proof: In Appendix I.

6. Comparison of the Models, and Some Data

We have, now, three possible models explaining endogenous growth. One is the Grossman-Helpman model, in which the innovation rate is given by

$$\iota = \frac{(1-1/\lambda)}{a_I} - \frac{\rho}{\lambda}.$$

Another is the Grossman-Helpman efficient solution, and here the innovation rate is given by

$$\iota^* = \frac{1}{a_I} - \frac{\rho}{\log \lambda}.$$

Finally, we have our model of competitive knowledge capital accumulation, in which the innovation rate is given by

$$j^* = \frac{b-\rho}{\log a}.$$

Each model is designed to get a closed form solution. Qualitatively, each is a similar function of the cost of innovating and the degree of impatience. As consumers are more patient, the frequency of innovation goes up. As it becomes more costly to innovate, innovation goes down. There are of course minor differences in the functional forms between these solutions. But the functional forms depend on a variety of assumptions – logarithmic utility, exponentially improving steps, and so forth – that were contrived to make the models easy to solve, so the particular functional forms have no strong claim of correctness. Moreover, the models differ in ways that are designed to ease the solution in each case.

What are then the substantive, as opposed to the technically convenient, differences? First, the parameter λ, how high each rung of the ladder is, has no effect in our model – this is due to the presence of two offsetting effects, increasing the intensity of innovation during the first build-up phase of the cycle and decreasing it during the second growth phase. However, this "neutrality" of step size in our model is due to the simplifications involved, and it is likely that various adjustments could yield either increasing innovation in step-size as in Grossman-Helpman, or decreasing innovation in step-size.

Second, the competitive innovation model has the extra parameter b, representing the rate at which productive capacity increases or is turned into usable output, possibly through imitation. As it becomes easier to reproduce knowledge capital, in the sense that b is larger, the intensity of innovation increases. In a loose sense the Grossman-Helpman model, like all models in which knowledge is a public good, assumes that $b = \infty$. This is because once the fixed cost is paid and a new rung has been introduced anyone can make an infinite number of copies of the jth good. A finite number of copies are produced only because the monopoly power of the first innovator is used to prevent additional entry. Put it differently, the standard model assumes that the movement from one particular vintage to the next can be infinitely rapid (once discovered, knowledge is a public good everyone can use); the presence of a fixed cost at each step explains why things do not go haywire.

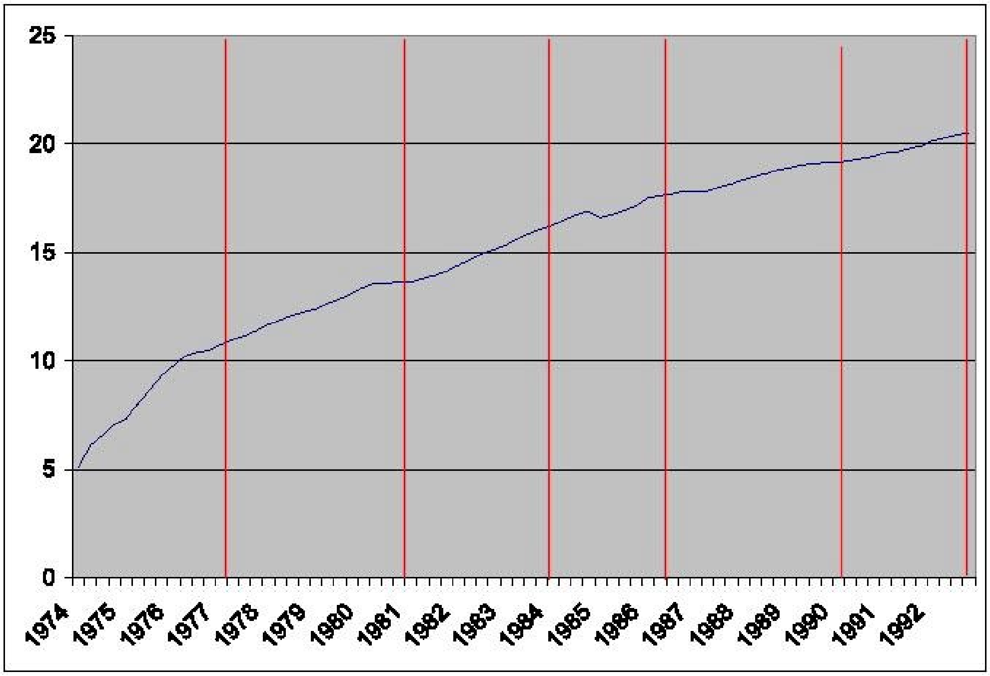
Finally, the model of competitive knowledge capital accumulation has a distinctive implication for the growth of output: it should grow in bursts (the growth phase) punctuated by periods in which output does not grow (the build-up phase). Moreover, the build-up phase should end when a new vintage of consumption is produced for the first time. Notice first that, per-se, the fact that the build-up phase has a positive length is not essential to our theory: when $a = \lambda$ we have that $\tau^b = 0$ and the new knowledge is used to produce consumption right away. Still, as long as $b > \lambda / a$ meaning that innovation is costlier than imitation, the former takes place only when all labor is employed with the latest available quality of knowledge. Further, the growth process still alternates between widening and deepening even if the latter is instantaneous and consumption grows at a constant balanced rate, as in most growth models. In this special case, our model "endogenizes" Solow's growth model of vintage capital with exogenous TFP growth.

Turning to the DRAM data from Irwin and Klenow [1994] we find that new vintages in fact trickle into production which is at variance with the model. This is not terribly surprising, as in practice even during the buildup phase there will be some demand for larger capacity chips for specialty use, and some chips produced for test purposes and subsequently sold for specialty use.

So let us take as our operational definition of "produced for the first time" the first time the new generation constitutes 5% of the total market for memory. We can then examine aggregate output of memory (that is measured in bytes, not chips) and plot this against the dates at which new generations of chips are introduced.

The Figure 4 shows this aggregate memory output as the blue curve, with the dates at which new generations are introduced as vertical red lines. The data does not show that every slowdown is followed by a switchover, nor even that every switchover is preceded by a slowdown (the 1984 switchover clearly is not), but rather that, in general, there are period of growth alternating with slowdowns. Remarkably these slowdowns have essentially zero growth as the theory predicts, and they are clearly associated with the switchovers. The ratio between the growth and buildup phases in the data is interesting too, as the buildup phases are very short, suggesting that λ is not much smaller than a.

Figure 4: Aggregate DRAM Output



7. Fixed Cost of Knowledge Capital

A key feature of this model is the build-up phase during which consumption is flat, and capital is accumulated for the "big push" to the next level of knowledge capital. Because the exact time at which conversion between qualities of knowledge capital takes place is indeterminate, we will focus on the equilibrium in which conversion does not take place until the new type of knowledge is needed – at the end of the build-up – at which time all current knowledge capital not needed for producing consumption is converted to the new level. In Appendix 2 we show that, in this equilibrium, the amount of current knowledge capital that will be converted is exactly

$$F^* = \frac{a^{\frac{\rho}{b-\rho}}}{a^{\frac{\rho}{b-\rho}}-1}\frac{\lambda^{\frac{\rho}{b-\rho}}-1}{\lambda^{\frac{\rho}{b-\rho}}}\frac{a-1}{\lambda-1}\frac{b-\rho}{\rho}.$$

What does such an equilibrium look like to an observer of the economy? She will notice that knowledge capital of the current type is cumulated beyond the amount needed to produce consumption until one day a new product is released, and a discrete investment is made, converting F^* units of old knowledge capital to new knowledge capital. Such an observer might be forgiven for suspecting that this is evidence of increasing returns to scale, a fixed cost in the production of knowledge, or an indivisibility in the production of knowledge. Is this inference warranted? Not necessarily: our model is one of purely diminishing returns to scale, and there are equilibria that exhibit exactly this behavior.

On the other hand, it is certainly plausible to argue that there is such a fixed cost. Suppose that there is indeed a fixed cost of introducing new knowledge capital for the first time. Specifically, to produce for the first time quality $j+1$ knowledge from quality j requires a fixed cost of F units of quality j knowledge. This results in the creation of $\bar{k} < F$ initial units of quality $j+1$ knowledge.[15] For computational simplicity and notational convenience, once the fixed cost is incurred, we assume that it is possible to convert additional units of quality j knowledge to quality $j+1$ knowledge at the same rate $F/\bar{k} = a$. This may seem a strong assumption – it might seem plausible that, after knowledge of a given quality is first converted to new knowledge capital, the cost of converting additional old knowledge capital will fall. It will become clear later that, properly modeled, such an assumption adds complication to the model, without changing the essential results.

The key observation is a simple one: if $F \le F^*$ – that is to say, if the indivisibility or fixed cost is not too large, then the presence of a competitive equilibrium is consistent with the presence of the fixed cost. Note that this is not a result of smoothing over a continuum of consumers as is the case with consumption

[15] As discussed in footnote 7, if it is possible to convert new knowledge capital to old at the same ratio as forward conversion, then fixed cost does not matter, as it can be “undone” by a subsequent backward conversion. This may partially explain why, when we consider the empirics of fixed costs, they do not seem to matter a great deal: most electrical engineers, if not most economists, can still change a light bulb by themselves.

indivisibilities: this is purely due to the way in which equilibrium production is arranged. Competitive equilibrium in the ordinary sense exists in the presence of a fixed cost.

On the other hand if $F > F^*$ then competitive equilibrium in the usual sense does not exist. There will be moments of time in which at competitive equilibrium prices it appears profitable to incur the fixed cost, while if it is actually incurred, prices will drop so much as to make it in fact unprofitable. This may be thought of as the classical case, the one increasing returns theorists have in mind.

8. Robustness

We have developed a model based on a number of special assumptions designed for ease of exposition and solution. We now verify that the basic conclusions are not sensitive to the special assumptions.

Features of the innovation process

First is the fact that when new knowledge capital is created from old, less of the new knowledge capital is made available. In other words, an additional unit of new knowledge requires more than one unit of the old knowledge. This is a natural assumption: the stock of knowledge capital is measured by the amount of labor it can employ and any process of "upgrading" – be it of machines, humans, or both – will entail the destruction of some previous employment capacity. This is, after all, what the popular expression "creative destruction" is meant to capture.

Second, there is an aspect of the conversion technology that may strike the reader as unrealistic: the conversion takes place instantly – while in fact the development of new knowledge generally takes time. This however turns out to be a matter of convenience rather than substance. Suppose that a units of quality j knowledge capital converted at time t do not become available as a unit of quality $j+1$ knowledge capital until time $t+\Delta$. Since the model is one of perfect foresight, the need for conversion can be foreseen, so if we imagine the conversion taking place at time $t+\Delta$ then the a units of quality j knowledge capital not being used to produce consumption would have grown to $ae^{b\Delta}$. Hence if we consider a model with no delay but cost of conversion $a' = ae^{b\Delta}$, this is equivalent to the model with delay and cost of conversion a. In effect, all we have done is to assume that any time delay is capitalized into the cost of conversion.

Labor Cost of Producing Knowledge

We have assumed that labor is not an input into the knowledge creation process. On the one hand, we doubt that the type of labor used in knowledge creation is a particularly good substitute for the labor used to produce output, so we do not view the alternative assumption as especially realistic either. On the other hand: what happens if we require some sort of labor or other input into the knowledge creation process? In the divisible case, it does not matter: the welfare theorems hold regardless of the details of the production process. In the fixed cost case, this creates an additional incentive to innovate gradually so as to avoid dragging labor out of the production of consumption all at once. Still, the sensible model would be one in which new knowledge accumulates gradually, but is not useful until a threshold is crossed. That is, the right place to put the constraint is in $\bar{k}$ and not in F. This just means that knowledge capital is not employed in producing output until after $\bar{k}$ units are acquired – either way, our analysis would not change in an important way.

Spillover Externalities

With a spillover externality the producer of knowledge capital does not own all the knowledge capital that he produces: some of it spills over onto other traders who get it for "free." In the extreme it all spills over, and the original producer gets nothing, but this is not a terribly interesting model. In practice the original innovator gets some share of the knowledge capital produced, share which may be larger or smaller depending on circumstances: no doubt in practice there is always some spillover. This will delay innovation even in the perfectly divisible case, as the innovator will receive only a portion of the marginal social benefit of his innovation rather than all of it. The resulting equilibrium will not be first best of course; however, innovation will still take place, it will simply take longer before the price of inputs drops sufficiently to make it profitable to innovate.

Cost of Converting Old Knowledge

We have assumed that after the initial fixed cost is paid the cost of converting further units of old knowledge into new ones remains unchanged. If we assume that the cost of conversion drops discretely after the initial unit of knowledge capital is produced, this is equivalent to a 100% spillover externality and no innovation will take place. However, that assumption makes little economic sense. We can distinguish two techniques for learning how to convert old knowledge into new: learning from scratch, and being taught. The former does not get easier just because somebody already did it. The latter requires the new knowledge as input. This suggests that the correct model is one with an additional activity: one that uses new knowledge to convert old knowledge to new. Consider first the divisible case. Intuitively the social optimum is to create a small amount of new knowledge using the learning from scratch technique, then switch over to the conversion activity. Firms that do this are called research universities. The key point is that, in this case as in our simpler model, it is not socially optimal to create an infinitesimal amount of new knowledge from scratch and then switch over right away. This would provide very little capacity for conversion. Rather there is some threshold in the amount of learning by scratch that is desirable before switching. This, then, is similar to our earlier analysis: the divisible case satisfies the welfare theorems as it simply has an additional activity. Since the divisible equilibrium involves accumulating a decent amount of knowledge using the "from scratch" activity before switching to the "conversion" activity, an indivisibility requiring a certain amount of "from scratch" learning need not bind.[16]

9. Conclusion

We give an account of endogenous growth and innovation in a world of purely decreasing returns. In our account competitive pressure and first mover advantage are at center stage. Although this economy is completely classical, satisfying the first and second welfare theorems, it never-the-less exhibits an endogenous rate of innovative activity. Innovation takes place in waves separated by pauses during which knowledge is accumulated for the next wave. In short, our economy displays endogenous innovation cycles around a long run trend.

Also, despite the fact that we study an economy with perfect divisibility and constant returns, to an external observer the sudden introduction of substantial amounts of new knowledge would suggest that there are fixed costs or increasing returns.

[16] The interested reader should consult Boldrin and Levine [2009b], where a model of discovery with exactly these properties is developed and analyzed.

Strikingly, this lumpy introduction of new knowledge implies that the equilibrium we study is robust with respect to the introduction of a technological indivisibility or a fixed cost, at least as long as the latter is not particularly large.

Finally: the model has implications for the controversial question of intellectual property. The conventional wisdom is that intellectual property is needed to encourage innovation, and that it is worth suffering the monopoly and social distortions from a modest level of protection. In this world that cannot be the case: since the equilibrium is first best, the monopoly and social distortions from intellectual property serve only to reduce the efficiency of the competitive innovation process.

Appendix 1: Propositions 2 and 3

Proposition 2: *During full employment no more than two qualities of knowledge capital are actually used to produce consumption, and these must be consecutive qualities. If* j' *is used to produce consumption*

$$q_{jt} \geq v_{jt}^{j'} \equiv \frac{\lambda^j - \lambda^{j'}}{b(1 - 1/a^{j-j'})} \frac{1}{c_t},$$

with equality if j *is also used to produce consumption. If two qualities are both used in positive amounts in producing consumption the latter grows at the rate* $b - \rho$.

Proof: The easiest way to do the computation is to imagine that newly freed quality j' knowledge is converted immediately to quality j. Consequently, to add one unit of quality j knowledge to the production of consumption requires drawing less than one unit of quality j knowledge away from other uses.

Specifically, increasing the quantity of quality j knowledge used in the production of consumption by $\varepsilon/(1 - 1/a^{j-j'})$ displaces the identical amount of quality j' knowledge. These $\varepsilon/(1 - 1/a^{j-j'})$ free units of quality j' knowledge convert to

$$1/a^{j-j'} \times \varepsilon/(1 - 1/a^{j-j'})$$

units of quality j knowledge. The net input of quality j knowledge needed to increase the amount used in consumption by $\varepsilon/(1 - 1/a^{j-j'})$ is the difference between the total requirement $\varepsilon/(1 - 1/a^{j-j'})$ and this extra quality j knowledge generated from freed knowledge j' capital

$$\frac{\varepsilon}{1-1/a^{j-j'}} - \frac{1}{a^{j-j'}}\frac{\varepsilon}{1-1/a^{j-j'}} = \varepsilon .$$

In other words, if we move ε units of quality j knowledge to displace quality j' in the production of consumption, the amount of quality j knowledge used in producing consumption increases by the greater amount $\varepsilon/(1-1/a^{j-j'})$. Another way of seeing this is: investing ε additional units of quality j knowledge capital in the production of consumption allows to shift $\varepsilon/(1-1/a^{j-j'})$ units of labor away from quality j' and toward quality j, the total installed capacity of which increases by $\varepsilon/(1-1/a^{j-j'})$.

We conclude that, over a short period of time τ, using an additional ε units of quality j knowledge capital to displace the inferior quality j' in the production of consumption increases consumption by

$$(\lambda^j - \lambda^{j'})\frac{\varepsilon}{1-1/a^{j-j'}}\tau ,$$

where $\lambda^j - \lambda^{j'}$ is the productivity differential, measured in consumption units, between the high and low quality knowledge.

Turning next to the use of knowledge in reproducing itself, as was the case in the initial unemployment phase, if quality j knowledge capital is used to reproduce itself, it yields $b\varepsilon\tau$ new units.

Putting together the two uses of knowledge in the full employment case, we conclude that one unit of j knowledge capital is a perfect substitute for

$$\frac{\lambda^j - \lambda^{j'}}{b(1-1/a^{j-j'})}$$

units of consumption, when knowledge of type j' is also used to produce consumption. This gives the marginal social value of a unit of quality j knowledge in producing consumption by displacing quality j' knowledge as

$$v_{jt}^{j'} = \frac{\lambda^j - \lambda^{j'}}{b(1-1/a^{j-j'})}\frac{1}{c_t}.$$

The price q_{jt} of quality j knowledge capital cannot be lower than $v_{jt}^{j'}$, since it cannot be strictly profitable to buy knowledge of type j and shift it into the production of consumption. So if j' is used to produce consumption

$$q_{jt} \geq v_{jt}^{j'},$$

with equality if j is also used to produce consumption.

Let us now show that only two qualities of knowledge capital are actually used to produce consumption and, when this is the case, consumption grows at the rate $b-\rho$. We make use of the following calculation:

$$\begin{aligned} v_{jt}^{j'} &= \frac{\lambda^{j}-\lambda^{j'}}{b(1-1/a^{j-j'})}\frac{1}{c_t} = a^{j-j'}\lambda^{j'}\frac{\lambda^{j-j'}-1}{a^{j-j'}-1}\frac{1}{bc_t} \\ &= a^{j-j'}\lambda^{j'}\frac{\lambda^{j-j'-1}+\lambda^{j-j'-2}\dots+1}{a^{j-j'-1}+a^{j-j'-2}\dots+1}\frac{\lambda-1}{a-1}\frac{1}{bc_t} \\ &< a^{j-j'}\lambda^{j'}\frac{\lambda-1}{a-1}\frac{1}{bc_t} = a^{j-j'-1}\left[a\lambda^{j'}\frac{\lambda-1}{a-1}\frac{1}{bc_t}\right] = a^{j-j'-1}v_{j'+1,t}^{j'} \end{aligned}.$$

Now suppose that quality j' and quality $j > j'+1$ are both used to produce consumption. Then $q_{jt} = v_{jt}^{j'}$. Zero profits on deepening implies $q_{jt} = a^{j-j'-1}q_{j'+1,t}$. Using $q_{j'+1,t} \geq v_{j'+1,t}^{j'}$, we have

$$a^{j-j'-1}v_{j'+1,t}^{j'} \leq a^{j-j'-1}q_{j'+1,t} = q_{jt} = v_{jt}^{j'} < a^{j-j'-1}v_{j'+1,t}^{j'},$$

the last step following from the inequality above. This contradiction establishes that if j' is used to produce consumption, then no higher quality than $j'+1$ may be used. In other words, at most two adjacent qualities of knowledge capital are used to produce consumption at any given point in time. When this is the case, we must have, for quality j, the price $q_{jt} = v_{jt}^{j-1}$. In particular, $\dot{v}_{jt}^{j-1}/v_{jt}^{j-1} = \dot{q}_{jt}/q_{jt} = -(b-\rho)$. Plugging in for v_{jt}^{j-1} this implies once again that consumption grows at $b-\rho$.

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Proposition 3: *Consumption remains constant during a build-up phase that lasts for*

$$\tau^{b} = \frac{\log a - \log\lambda}{b-\rho},$$

followed by a growth phase during which consumption grows at the rate $b-\rho$ lasting

$$\tau^{g} = \frac{\log\lambda}{b-\rho}$$

The total length of a cycle is

$$\tau^* = \frac{\log a}{b - \rho}$$

so that the resulting research intensity is

$$1 / \tau^* = j* = \frac{b - \rho}{\log a}$$

Proof: We may assume without loss of generality that the initial quality used is $j = 0$. If the unemployment phase ends at t, then $c_t = 1$. The price is $q_{0t} = v_{0t} = 1 / b$, from which we can see that the price of quality $j = 1$ knowledge is $q_{1t} = a / b$ at the time it is introduced. The value of this knowledge capital in producing consumption is

$$v_{1t}^0 = \frac{\lambda - 1}{b(1 - 1/a)} = \frac{a}{b}\frac{\lambda - 1}{a - 1} < q_{1t} .$$

That is, right when it is first produced, it does not pay to use quality $j = 1$ knowledge to produce consumption. It follows that consumption must now remain constant for some time; as long as consumption remains constant, so does the value of knowledge of quality $j = 1$ in producing it, $v_{1,t+\tau}^0 = v_{1t}^0$. On the other hand, the price of quality $j = 1$ knowledge capital is falling as it accumulates over time, so that, at $t + \tau$ it is $q_{1,t+\tau} = (a / b)e^{-(b-\rho)\tau}$. When

$$v_{1t}^0 = \frac{a}{b}\frac{\lambda - 1}{a - 1} = (a / b)e^{-(b-\rho)\tau} = q_{1t}$$

that is, at

$$\tau = \frac{1}{b - \rho}\log\frac{a - 1}{\lambda - 1}$$

it becomes profitable to introduce quality $j = 1$ knowledge into producing consumption, and consumption then grows at the rate $b - \rho$.

In general, when quality j knowledge capital is first created quality adjusted consumption is λ^{j-1} and further increases require quality j to be used in its production until consumption reaches λ^j. During the growth phase, qualities $j - 1$ and j are used, and consumption grows at $b - \rho$. Hence the length of the growth phase τ^g is characterized by

$$\lambda^{j-1}e^{(b-\rho)\tau^g} = \lambda^j$$

meaning that

$$\tau^g = \frac{\log \lambda}{b - \rho}.$$

Applying to the general case the argument developed earlier for qualities $j = 0$ and $j = 1$, we see that, at the end of the j-th growth phase, the price of quality $j + 1$ knowledge capital must be

$$q_{j+1,t} = aq_{jt} = av_{jt}^{j-1} = a\frac{\lambda^j - \lambda^{j-1}}{b(1 - 1/a)}\frac{1}{\lambda^j} = \frac{\lambda - 1}{b(1 - 1/a)}\frac{a}{\lambda}.$$

The consumption value of quality $j + 1$ knowledge capital is

$$v_{j+1,t}^{j} = \frac{\lambda^{j+1} - \lambda^j}{b(1 - 1/a)}\frac{1}{\lambda^j} = \frac{\lambda - 1}{b(1 - 1/a)},$$

implying, again, that $v_{j+1,t}^{j} < q_{j+1,t}$. In other words, at the end of each *growth phase*, we must have a *build-up phase*, during which consumption remains fixed at λ^j, while the price of quality $j + 1$ knowledge (which is being accumulated) falls by a factor of λ / a. Since it falls at the constant rate $b - \rho$, this takes

$$\tau^b = \frac{\log a - \log \lambda}{b - \rho}.$$

Following this, we again begin the growth phase, and the cycle repeats at the next level of the quality ladder.

The total length of the cycle τ^* follows from adding $\tau^g + \tau^b$, and the research intensity – the rate at which we move up the ladder – is by definition is just the inverse of the cycle length.

☑

Appendix 2: Capital Levels

Consider a growth phase beginning at time 0. Suppose that initial knowledge capital used in consumption is entirely of quality j, and that there are k_{j0} units of that quality. Suppose that consumption starts at c_0, possibly through an initial jump. It will be convenient to define this jump as $\xi_0 = c_0 / \lambda^j$, with $1 < \xi_0 < \lambda$. At time t consumption has grown to $c_t = c_0 e^{(b-\rho)t}$ until it reaches λ^{j+1} ending the growth phase at time

$$t = \tau^g = \frac{\log(\lambda) - \log(\xi_0)}{b - \rho}.$$

During this time we assume that quality j knowledge capital is converted to quality $j+1$ as soon as it is freed from use in producing consumption. Specifically,

$$\begin{aligned} k_{jt} = d_{jt} &= \frac{\lambda^{j+1} - c_t}{\lambda^{j+1} - \lambda^j} = \frac{\lambda^{j+1} - \lambda^j \xi_0 e^{(b-\rho)t}}{\lambda^{j+1} - \lambda^j} \\ &= \frac{\lambda - \xi_0 e^{(b-\rho)t}}{\lambda - 1} \end{aligned}.$$

So

$$\begin{aligned} h_{jt} &= -\dot{k}_{jt} = -\dot{d}_{jt} \\ &= \frac{(\dot{c}_t / \lambda^j)}{\lambda - 1} = (b - \rho)\frac{(c_t / \lambda^j)}{\lambda - 1} \\ &= \frac{\xi_0 (b - \rho)}{\lambda - 1} e^{(b-\rho)t} \end{aligned}$$

We find then that quality $j+1$ knowledge capital grows according to

$$\begin{aligned} \dot{k}_{j+1,t} &= b(k_{j+1,t} + d_{jt} - 1) + \frac{\xi_0 (b - \rho)}{a(\lambda - 1)} e^{(b-\rho)t} \\ &= b(k_{j+1,t} + \frac{1 - \xi_0 e^{(b-\rho)t}}{\lambda - 1}) + \frac{\xi_0 (b - \rho)}{a(\lambda - 1)} e^{(b-\rho)t} \\ &= b(k_{j+1,t} + \frac{1}{\lambda - 1}) + \frac{(1-a)b - \rho}{a} e^{(b-\rho)t} \frac{\xi_0}{\lambda - 1} \end{aligned}$$

until τ^g.

The build-up phase comes next and lasts until $t = \tau^g + \tau^b$, at which time quality $j+1$ knowledge is $k_{j+1,t} = 1 + e^{b\tau^b} \left(k_{j+1,\tau^g} - 1\right)$. Since $\tau^g + \tau^b = \log a / (b - \rho)$, we have, from above, that

$$\tau^b = \frac{\log(a) - \log(\lambda) + \log(\xi_0)}{b - \rho}, \text{ or}$$

$$k_{j+1,\tau^g + \tau^b} = 1 + \left(\frac{a\xi_0}{\lambda}\right)^{\frac{b}{b-\rho}} \left(k_{j+1,\tau^g} - 1\right).$$

We next guess a solution to

$$\dot{k}_{j+1,t} = b(k_{j+1,t} + \frac{1}{\lambda - 1}) + \frac{(1-a)b - \rho}{a} e^{(b-\rho)t} \frac{\xi_0}{\lambda - 1}$$

of the form $k_{j+1,t} = a_0 + a_1 e^{bt} + a_2 e^{(b-\rho)t}$. By differentiating

$$\dot{k}_{j+1,t} = b a_1 e^{bt} + (b - \rho) a_2 e^{(b-\rho)t} \ ,$$

substituting into the differential equation gives

$$\begin{aligned} \dot{k}_{j+1,t} &= b(a_0 + a_1 e^{bt} + a_2 e^{(b-\rho)t} + \frac{1}{\lambda - 1}) + \frac{(1-a)b - \rho}{a} e^{(b-\rho)t} \frac{\xi_0}{\lambda - 1} \\ &= b\left(a_0 + \frac{1}{\lambda - 1}\right) + b a_1 e^{bt} + \left(b a_2 + \frac{(1-a)b - \rho}{a} \frac{\xi_0}{\lambda - 1}\right) e^{(b-\rho)t} \end{aligned}.$$

We conclude that

$$a_0 + \frac{1}{\lambda - 1} = 0$$

and

$$(b - \rho) a_2 = b a_2 + \frac{(1-a)b - \rho}{a} \frac{\xi_0}{\lambda - 1}.$$

This gives

$$a_0 = -1/(\lambda - 1)$$

and

$$a_2 = \frac{(a-1)b + \rho}{a\rho} \frac{\xi_0}{\lambda - 1}.$$

Plugging back into our guessed solution $k_{j+1,t} = a_0 + a_1 e^{bt} + a_2 e^{(b-\rho)t}$

$$k_{j+1,t} = -\frac{1}{\lambda - 1} + a_1 e^{bt} + \frac{(a-1)b + \rho}{a\rho} \frac{\xi_0}{\lambda - 1} e^{(b-\rho)t}.$$

The initial condition is

$$k_{j+1,0} = \frac{k_{j0} - d_{j0}}{a} = \frac{k_{j0} - \dfrac{\lambda - \xi_0}{\lambda - 1}}{a},$$

hence

$$\frac{k_{j0} - \dfrac{\lambda - \xi_0}{\lambda - 1}}{a} = -\frac{1}{\lambda - 1} + a_1 + \frac{(a-1)b + \rho}{a\rho} \frac{\xi_0}{\lambda - 1},$$

enabling us to find the missing coefficient

$$a_1 = \frac{k_{j0} - \dfrac{\lambda - \xi_0}{\lambda - 1}}{a} + \frac{1}{\lambda - 1} + \frac{(1-a)b - \rho}{\rho a}\frac{\xi_0}{\lambda - 1}.$$

Consequently

$$k_{j+1,t} =$$

$$-\frac{1}{\lambda-1} + \left(\frac{k_{j0} - \dfrac{\lambda - \xi_0}{\lambda - 1}}{a} + \frac{1}{\lambda - 1} + \frac{(1-a)b - \rho}{\rho a}\frac{\xi_0}{\lambda - 1}\right)e^{bt} + \frac{(a-1)b + \rho}{a\rho}\frac{\xi_0}{\lambda - 1}e^{(b-\rho)t}$$

Recall that the growth phase lasts for

$$\tau^g = \frac{\log(\lambda) - \log(\xi_0)}{b - \rho},$$

so $e^{(b-\rho)\tau^g} = \lambda / \xi_0$ and

$$e^{b\tau^g} = \left(\frac{\lambda}{\xi_0}\right)^{\frac{b}{b-\rho}}.$$

Plugging in we find

$$k_{j+1,\tau^g} \quad -\frac{1}{\lambda-1} \quad \frac{k_{j0} - \dfrac{\lambda - \xi_0}{\lambda - 1}}{a} \quad \frac{1}{\lambda - 1} \quad \frac{(1-a)b - \rho}{\rho a}\frac{\xi_0}{\lambda - 1} \quad \frac{\lambda}{\xi_0}^{\frac{b}{b-\rho}}$$

$$\frac{(a-1)b \quad \rho}{\rho a}\frac{\xi_0}{\lambda - 1}\frac{\lambda}{\xi_0}$$

$$-\frac{1}{\lambda-1} \quad \frac{k_{j0} - \dfrac{\lambda - \xi_0}{\lambda - 1}}{a} \quad \frac{1}{\lambda - 1} \quad \frac{(1-a)b - \rho}{\rho a}\frac{\xi_0}{\lambda - 1} \quad \frac{\lambda}{\xi_0}^{\frac{b}{b-\rho}}$$

$$\frac{(a-1)b \quad \rho}{\rho a}\frac{\lambda}{\lambda - 1}$$

while, from above, the terminal stock of knowledge capital is

$$
\begin{aligned}
k_{j+1,\tau^g+\tau^b} &= 1+\left(\frac{a\xi_0}{\lambda}\right)^{\frac{b}{b-\rho}}(k_{j+1,\tau^g}-1)\\
&= 1+\left(\frac{a\xi_0}{\lambda}\right)^{\frac{b}{b-\rho}}\times\\
&\left\{\begin{array}{l}-\dfrac{1}{\lambda-1}+\left(\dfrac{k_{j0}-\dfrac{\lambda-\xi_0}{\lambda-1}}{a}+\dfrac{1}{\lambda-1}+\dfrac{(1-a)b-\rho}{\rho a}\dfrac{\xi_0}{\lambda-1}\right)\left(\dfrac{\lambda}{\xi_0}\right)^{\frac{b}{b-\rho}}\\
+\dfrac{(a-1)b+\rho}{\rho a}\dfrac{\lambda}{\lambda-1}-1\end{array}\right\}\\
&= 1+\left(\frac{a\xi_0}{\lambda}\right)^{\frac{b}{b-\rho}}\times\\
&\left\{\begin{array}{l}-\dfrac{\lambda}{\lambda-1}+\left(\dfrac{k_{j0}-\dfrac{\lambda-\xi_0}{\lambda-1}}{a}+\dfrac{1}{\lambda-1}+\dfrac{(1-a)b-\rho}{\rho a}\dfrac{\xi_0}{\lambda-1}\right)\left(\dfrac{\lambda}{\xi_0}\right)^{\frac{b}{b-\rho}}\\
+\dfrac{(a-1)b+\rho}{\rho a}\dfrac{\lambda}{\lambda-1}\end{array}\right\},
\end{aligned}
$$

$$
\begin{aligned}
k_{j+1,\tau^b} &= 1+\\
&-\frac{\lambda}{\lambda-1}\left(\frac{a\xi_0}{\lambda}\right)^{\frac{b}{b-\rho}}+\left(\frac{k_{j0}-\dfrac{\lambda-\xi_0}{\lambda-1}}{a}+\frac{1}{\lambda-1}+\frac{(1-a)b-\rho}{\rho a}\frac{\xi_0}{\lambda-1}\right)a^{\frac{b}{b-\rho}}\\
&+\frac{(a-1)b+\rho}{\rho a}\frac{\lambda}{\lambda-1}\left(\frac{a\xi_0}{\lambda}\right)^{\frac{b}{b-\rho}}\\
&= 1+\frac{1}{\lambda-1}a^{\frac{b}{b-\rho}}-\frac{\lambda}{\lambda-1}a^{\frac{\rho}{b-\rho}}\\
&-\frac{\lambda}{\lambda-1}\left(\frac{a\xi_0}{\lambda}\right)^{\frac{b}{b-\rho}}+\left(\frac{\xi_0}{\lambda-1}+\frac{(1-a)b-\rho}{\rho}\frac{\xi_0}{\lambda-1}\right)a^{\frac{\rho}{b-\rho}}\\
&+\frac{(a-1)b+\rho}{\rho a}\frac{\lambda}{\lambda-1}\left(\frac{a\xi_0}{\lambda}\right)^{\frac{b}{b-\rho}}\\
&+a^{\frac{\rho}{b-\rho}}k_{j0}
\end{aligned}
$$

$$
\begin{aligned}
k_{j+1,\tau^b} &= 1+\frac{1}{\lambda-1}\left(a^{\frac{b}{b-\rho}}-\lambda a^{\frac{\rho}{b-\rho}}\right) \\
&\quad -\left(1+\frac{(1-a)b-\rho}{\rho a}\right)\frac{\lambda}{\lambda-1}\left(\frac{a\xi_0}{\lambda}\right)^{\frac{b}{b-\rho}} \\
&\quad +\left(1+\frac{(1-a)b-\rho}{\rho}\right)\frac{a^{\frac{\rho}{b-\rho}}}{\lambda-1}\xi_0 \\
&\quad +a^{\frac{\rho}{b-\rho}}k_{j0} \\
&= 1+\frac{1}{\lambda-1}\left(a^{\frac{b}{b-\rho}}-\lambda a^{\frac{\rho}{b-\rho}}\right) \\
&\quad +\frac{(a-1)(b-\rho)}{\rho}\frac{\lambda a^{\frac{\rho}{b-\rho}}}{\lambda-1}\left(\frac{\xi_0}{\lambda}\right)^{\frac{b}{b-\rho}}+\frac{(1-a)b}{\rho}\frac{a^{\frac{\rho}{b-\rho}}}{\lambda-1}\xi_0 \\
&\quad +a^{\frac{\rho}{b-\rho}}k_{j0}
\end{aligned}
$$

$$
\begin{aligned}
k_{j+1,\tau^b} &= 1+\frac{1}{\lambda-1}\left(a^{\frac{b}{b-\rho}}-\lambda a^{\frac{\rho}{b-\rho}}\right) \\
&\quad +\frac{a-1}{\rho}\frac{a^{\frac{\rho}{b-\rho}}}{\lambda-1}\left((b-\rho)\lambda\left(\frac{\xi_0}{\lambda}\right)^{\frac{b}{b-\rho}}-b\xi_0\right) \\
&\quad +a^{\frac{\rho}{b-\rho}}k_{j0}
\end{aligned}
$$

So the steady state condition is $k_{j+1,t^b}=k_{j0}$, or

$$
\begin{aligned}
\left(a^{\frac{\rho}{b-\rho}}-1\right)k_{j0} &= -1-\frac{1}{\lambda-1}\left(a^{\frac{b}{b-\rho}}-\lambda a^{\frac{\rho}{b-\rho}}\right) \\
&\quad -\frac{a-1}{\rho}\frac{a^{\frac{\rho}{b-\rho}}}{\lambda-1}\left((b-\rho)\lambda\left(\frac{\xi_0}{\lambda}\right)^{\frac{b}{b-\rho}}-b\xi_0\right)
\end{aligned}
$$

Note that this function is increasing in ξ_0, specifically,

$$\left(a^{\frac{\rho}{b-\rho}}-1\right)\frac{dk_{j0}}{d\xi_0}=-\frac{a-1}{\rho}\frac{a^{\frac{\rho}{b-\rho}}}{\lambda-1}\left(b\lambda^{-\frac{\rho}{b-\rho}}\xi_0^{\frac{\rho}{b-\rho}}-b\right)$$

$$=-\frac{a-1}{\rho}\frac{a^{\frac{\rho}{b-\rho}}}{\lambda-1}b\left(\left(\frac{\xi_0}{\lambda}\right)^{\frac{\rho}{b-\rho}}-1\right)$$

It follows that

$$F^*=k_{j0}-d_{j0}=k_{j0}-\frac{\lambda-\xi_0}{\lambda-1}=k_{j0}-\frac{\lambda}{\lambda-1}+\frac{\xi_0}{\lambda-1}$$

is also increasing in ξ_0. For reference we can write out the function

$$F^*=$$

$$\frac{-1-\frac{1}{\lambda-1}\left(a^{\frac{b}{b-\rho}}-\lambda a^{\frac{\rho}{b-\rho}}\right)-\frac{a-1}{\rho}\frac{a^{\frac{\rho}{b-\rho}}}{\lambda-1}\left((b-\rho)\lambda\left(\frac{\xi_0}{\lambda}\right)^{\frac{b}{b-\rho}}-b\tilde{c}_0\right)}{a^{\frac{\rho}{b-\rho}}-1}-\frac{\lambda-\xi_0}{\lambda-1}$$

The sanity check is when $\xi_0=\lambda$, meaning that there is no growth phase

$$\left(a^{\frac{\rho}{b-\rho}}-1\right)k_{j0}=-1-\frac{1}{\lambda-1}\left(a^{\frac{b}{b-\rho}}-\lambda a^{\frac{\rho}{b-\rho}}\right)$$

$$-\frac{a-1}{\rho}\frac{a^{\frac{\rho}{b-\rho}}}{\lambda-1}((b-\rho)\lambda-b\lambda)$$

$$=-1-\frac{1}{\lambda-1}\left(a^{\frac{b}{b-\rho}}-\lambda a^{\frac{\rho}{b-\rho}}\right)$$

$$+(a-1)\frac{\lambda a^{\frac{\rho}{b-\rho}}}{\lambda-1}$$

$$=-1-\frac{1}{\lambda-1}\left(a^{\frac{b}{b-\rho}}-\lambda a^{\frac{\rho}{b-\rho}}\right)$$

$$+\frac{\lambda a^{\frac{b}{b-\rho}}}{\lambda-1}-\frac{\lambda a^{\frac{\rho}{b-\rho}}}{\lambda-1}$$

$$\left(a^{\frac{\rho}{b-\rho}}-1\right)k_{j0}=-1-\frac{1}{\lambda-1}\left(a^{\frac{b}{b-\rho}}-\lambda a^{\frac{\rho}{b-\rho}}\right)$$

$$+\frac{\lambda a^{\frac{b}{b-\rho}}}{\lambda-1}-\frac{\lambda a^{\frac{\rho}{b-\rho}}}{\lambda-1}$$

$$=-1-\frac{a^{\frac{b}{b-\rho}}}{\lambda-1}+\frac{\lambda a^{\frac{b}{b-\rho}}}{\lambda-1}$$

$$=a^{\frac{b}{b-\rho}}-1$$

while the direct calculation is

$$1+\left((k_{j0}\,/\,a)-1\right)e^{bt^b}=k_{j0}$$

$$1+\left((k_{j0}\,/\,a)-1\right)a^{\frac{b}{b-\rho}}=k_{j0}$$

$$k_{j0}\left(a^{\frac{\rho}{b-\rho}}-1\right)=a^{\frac{b}{b-\rho}}-1$$

Finally, when $\xi_0=1$, that is, when there is no jump in consumption at the start of growth

$$F^*=$$

$$\frac{-1-\dfrac{1}{\lambda-1}\left(a^{\frac{b}{b-\rho}}-\lambda a^{\frac{\rho}{b-\rho}}\right)-\dfrac{a-1}{\rho}\dfrac{a^{\frac{\rho}{b-\rho}}}{\lambda-1}\left((b-\rho)\lambda^{-\frac{\rho}{b-\rho}}-b\right)}{a^{\frac{\rho}{b-\rho}}-1}-1=$$

$$\frac{a^{\frac{\rho}{b-\rho}}\dfrac{1}{\lambda-1}\left(-(a-\lambda)-\dfrac{a-1}{\rho}\left((b-\rho)\lambda^{-\frac{\rho}{b-\rho}}-b\right)\right)-1}{a^{\frac{\rho}{b-\rho}}-1}-1$$

$$F^* =$$

$$\frac{a^{\frac{\rho}{b-\rho}}\left[1+\frac{1}{\lambda-1}\left(-(a-\lambda)-\frac{a-1}{\rho}\left((b-\rho)\lambda^{-\frac{\rho}{b-\rho}}-b\right)\right)-1\right]-1}{a^{\frac{\rho}{b-\rho}}-1}-1=$$

$$\frac{a^{\frac{\rho}{b-\rho}}\left[1+\frac{1}{\lambda-1}\left(-(a-\lambda)-\frac{a-1}{\rho}\left((b-\rho)\lambda^{-\frac{\rho}{b-\rho}}-b\right)-\lambda+1\right)\right]-1}{a^{\frac{\rho}{b-\rho}}-1}-1=$$

$$\frac{a^{\frac{\rho}{b-\rho}}\left[1+\frac{1}{\lambda-1}\left(-(a-1)-\frac{a-1}{\rho}\left((b-\rho)\lambda^{-\frac{\rho}{b-\rho}}-b\right)\right)\right]-1}{a^{\frac{\rho}{b-\rho}}-1}-1=$$

$$\frac{a^{\frac{\rho}{b-\rho}}\left[1+\frac{a-1}{\lambda-1}\left(-\frac{1}{\rho}\left((b-\rho)\lambda^{-\frac{\rho}{b-\rho}}-b+\rho\right)\right)\right]-1}{a^{\frac{\rho}{b-\rho}}-1}-1=$$

$$F^* =$$

$$\frac{a^{\frac{\rho}{b-\rho}}\left[1+\frac{a-1}{\lambda-1}\frac{b-\rho}{\rho}\left(1-\lambda^{-\frac{\rho}{b-\rho}}\right)\right]-1}{a^{\frac{\rho}{b-\rho}}-1}-1=$$

$$\frac{a^{\frac{\rho}{b-\rho}}}{a^{\frac{\rho}{b-\rho}}-1}\frac{\lambda^{\frac{\rho}{b-\rho}}-1}{\lambda^{\frac{\rho}{b-\rho}}}\frac{a-1}{\lambda-1}\frac{b-\rho}{\rho}$$

which is strictly positive.

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