Production Chains

David K. Levine*

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Abstract: More advanced technologies demand higher degrees of specialization – and longer chains of production connecting raw inputs to final outputs. Longer production chains are subject to a "weakest link" effect: they are more fragile and more prone to failure. Optimal chain length is determined by the trade-off between the gains to specialization and the higher failure rate associated with longer chain length. There is a kind of reverse "Keynesian multiplier" that magnifies the effect of real shocks. Consequently, more advanced economies should have higher unemployment rates and be more prone to crisis. The implications of the theory both for measurement and government policy are examined.

* Department of Economics, Washington University in St. Louis

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1. Introduction

Since Adam Smith there has been little doubt that the wealth of nations is driven by the degree of specialization. The amount of specialization in a modern economy is remarkable – Seabright's [2004] description of the number of ingredients and countries involved in the production of an item so humble as a shirt is a beautiful illustration of this point. It is illustrated as well by the number of parts in a modern good: according to Boeing, for example, there are over 6,000,000 parts in a 747.

Specialization in turn implies many stages of production, an aspect of the production function not present in most economic models. The goal of this paper is to give a simple account of how the number of stages of production are determined – the determinants of the degree of specialization – and the implications for growth and development.

The key motivating idea is that long "chains" of production are vulnerable to failure of a single link. Hence while long chains permit a high degree of specialization and so a large quantity of output, they are also more prone to failure. Consequently, production chains form a natural basis for an economy in which there are low risk low reward projects and high risk high reward projects, a common notion in the study of financial market imperfections.

This idea that failure of a single link may cause a "cascade" of failures is not a new one. It is implicit in the idea of a Keynesian multiplier – employing or unemploying a single person reverberates through the economy much as the unemployment of a single person in a production chain unravels the entire chain. Similarly, Leijonhufvud's [1976] notion of the vulnerability of long chains of credit to a single bankruptcy has a flavor similar to the disruption of a long chain of production by the failure of a single producer. Moreover the spirit of the model here is similar to that of Becker and Murphy [1992] who also argue that the degree of specialization is limited to a large extent by the problem that long chains of production are more inclined to failure.

The notion of stages of production is not new either. In industrial organization short production chains have been studied in the context of vertical versus horizontal integration, such as, for example, in Grossman and Hart [1986], but the issue of chain fragility is not addressed. A part of the macroeconomics literature has studied economies made up of a single broad production chain. Such models have been used in international trade, for example by Dixit and Grossman [1982] and to study how the pricing consequences of monetary policy feed down the supply chain, for example by Huang and Liu [2001].

More relevant is the multi-sectoral work of Horvath [1998, 2000] who shows how independent shocks across sectors can be magnified by linkages across sectors. However, unlike the model here where there are many narrow parallel chains, the issue of fragility does not arise with a single broad chain. Horvath's work has been recently generalized by Acemoglu, Ozdaglar and Tahbaz-Salehi [2010] who examine many interacting sectors, addressing the question of when shocks spread across many sectors tend to average out as opposed to creating aggregate fluctuations. Grossman and Rossi-Hansberg [2008] appear to have long chains of fixed length in their study of off-shoring. Their production technology is difficult to infer from their paper, and their focus in any case is on the location of chains links, not the fragility of chains.

The issue of fragility is the key idea in Kremer's [1993] o-ring theory of development, but he focuses on assortive matching of workers in the presence of complementaries. Recently, no doubt inspired by the current crisis, there has been a resurgence of interest in the fragility of chains. Jones [2010] studies a broad intermediate goods sector producing many different kinds of inputs all used in intermediate goods production with a one-period lag. The distribution of productivity across sectors and the level of complementarities determine the multiplier effect of resource misallocation across sectors. It acts much like the parameter determining the correlation of shocks across chains does in this model. Two recent papers build on this work by adopting a network perspective. Carvalho [2010] advances Jones work by using data on direct inputs rather than just the input output table data. Oberfield [2011] traces the dynamics of shocks across sectors. In a sense the model in this paper simplifies matters by adopting a Leontief specification for technology. The advantage is that this makes it possible to address the depth of chains and the determination of the degree of specialization as well as the role of capital market imperfections.

In a sense the central question addressed by this paper is "Are capitalist economies prone to crises?" There is a long history of conjecture that this might be the case, running from Marx to Keynes. What the alternative to a "capitalist economy" is and whether this is an empirical fact have not to my knowledge been resolved. However, the notion of production chains leads to a more specific and testable conjecture – capitalist economies by virtue of being efficient employ longer and more specialized production chains than more poorly organized economies. This raises output, welfare and utility, but it also leads to more "crises" than shorter less efficient production chains in the sense that output is more volatile. One might summarize by saying: capitalist economies are more subject to crises than less efficient economies – and this is a good thing.

The model has several related implications. First, suppose the measure of the size of a real shock is the number of links that fail. The consequence of such a shock depends heavily on how the failures are distributed. If they are concentrated in particular chains, the resulting unemployment and output loss is small. If they are spread across many chains, the resulting unemployment and output loss is large. So the model is consistent with the idea that a small real shock may have either large or small consequences – the magnitude of a shock is not properly measured by the number of links that fail, but rather by the number of chains that fail.

We are also interested in the role of financial markets. We model a simple financial market imperfection where insurance is limited by repayment constraints as in Kehoe and Levine [1993]. Better insurance markets mean that the second best will have longer chains, and higher output and welfare. We examine also the role of government policy.

2. The Technology

There is an infinite sequence of different intermediate goods $j = 1, 2, \ldots$ and one final good 0. There is a continuum of individuals. Each individual is endowed with one unit of labor. Anyone can use x units of labor to produce βx units of any type of intermediate good or the final good.

Anybody can choose to be a specialist of any type. A specialist of type j produces $\lambda_j x$ units of $j-1$ from x units of j where $\lambda_j \geq 1$ and $j \geq 1$. Utility is only for consumption c of the final good, and is represented by a strictly increasing, strictly concave function and smooth function $u(c)$.

A k*-production chain* for $k \geq 1$ consists of one generalist and specialists $j = 1, \ldots k-1$. The generalist produces intermediate good $k-1$. The labor of the chain must be used exclusively within the chain, and the output of the chain is equally divided among chain members. Consequently, the per capita output of the chain is $f(k) = \Pi_{j=1}^{k-1} \lambda_j \beta$ if $k > 1$, and $f(1) = \beta$.

After a chain is formed and specialties chosen, there is a probability $0 \le p \le 1$ that each individual "fails." Any chain with one or more failed individuals is unable to produce any output. We imagine that the producer may be sick, be involved in an accident, have a specialized machine that breaks, go bankrupt, or otherwise be unavailable. The critical feature of the model is that if a chain has one failure then the entire chain cannot produce.

Critical to the analysis is the correlation of failures across chains. It may be that failures are independent across individuals. It may also be that the probability of failure is higher if another individual in the same chain fails: that is, shocks may be more likely to occur to individuals working in the same chain. Moreover, it may be possible to swap individuals between chains. If specialist j in one chain fails, it is natural to try to obtain a j specialist from another chain.

It is tempting here to think in terms of large mass markets forgetting social feasibility constraints. That is, if I am a car manufacturer and my tire supplier fails, I just go to another supplier to get tires. But in general equilibrium, the other supplier was supplying tires to someone – and if I get those tires, some other automobile manufacturer does not. However: if another automobile making chain has a piston maker who has failed, then the tire manufacturer in that chain is unemployed and happy to provide me with tires. Hence when there are failures, some reduction in output must be accepted, but it is best to try to concentrate the failures all in the same chain. A simple example with three chains of length three and three failures makes the point.

Suppose there are three chains producing cars: Jaguars, BMWs, and Toyotas. Each chain has three links: tires, pistons, and unspecialized. Suppose there are three failures. Consider three different patterns of failure: labeled good, intermediate and bad.

Type of shock		Jaguar	BMW	Toyota
good	Tires	x		
	Pistons	x		
	Unspecialized	x		
intermediate	Tires	x		
	Pistons		x	
	Unspecialized			x
bad	Tires	x	x	x
	Pistons			
	Unspecialized			

In the good case only the Jaguar chain fails. This is the best outcome – with three failures, at most two chains can produce output. In the bad case all the tire producers fail: in this case it is impossible to produce any cars. Finally, in the intermediate case one link of each type fails. If it is possible to reorganize the chains to move all the failures to one chain, then two chains can produce output. Whether this is possible depends on market organization, the availability of information, the quality of business connections, and the degree of substitutability between specialists. With a high degree of substitutability and good market organization, two chains may be able to produce. With poor substitutability or poor market organization, none of the chains will be able to produce.

We can think of the situation in terms of the *reliability* of a chain $R(k)$, that is, the probability a chain of length k succeeds, as well as the per capita output $y = f(k)$ of a chain of length k. If shocks are highly correlated, then $R(k)$ falls slowly with chain length. If shocks are largely independent, then $R(k)$ falls rapidly with chain length. Expected output is given by $R(k)f(k)$. Notice that it is easy to construct models in which expected output is maximized as $k \to \infty$: if, for example, if $f(k) \to \infty$ much more quickly than $R(k) \to 0$, or if $R(k)$ is bounded away from zero and $f(k)$ is not bounded away from infinity. However, such models predict that chains produce very rarely – that they have very high failure probabilities – but on those rare occasions when they do produce, they produce gigantic amounts. As we do not observe such technologies being used,[1] our goal in designing a model will be to choose functional forms for which this is not the case.

There are two simple models that capture a positive correlation of failures. One is to assume that there is a positive probability that an individual is in a chain consisting entirely of successes. This leads to a model in which the probability of failure is bounded away from zero independent of k, and as we observed, this leads to the implausible prediction that there is rarely any output, but occasionally there are bursts of exceptionally large levels of output. In addition, it seems unlikely that intermediate goods are perfect substitutes: while final goods, such as Jaguar automobiles and BMW automobiles may be good substitutes, the pistons for their engines are not.

Here we make the opposite assumption: we assume that there is a positive probability that an individual is in a chain consisting entirely of failures. Specifically, we assume that there is a probability $r \leq p$ that an individual is in such a chain; with probability $1-r$ she is in a chain in which each individual has an independent $(1-p)/(1-r)$ probability of success. The overall probability of failure for an individual is then $r+(1-r)(1-(1-p)/(1-r)) = p$, as desired. The resulting probability a chain succeeds is $R(k)=(1-r)((1-p)/(1-r))^k$.

Next, consider the production process. What happens if λ_j is constant? In this case as the length of the chain increases the probability of success goes to zero exponentially, and the output of the chain increases exponentially. Unless the exponential rates are exactly the same, the optimal chain length is either one or infinity. The former case is neither interesting nor empirically relevant. The latter case is the one we have just argued is also empirically irrelevant. Hence we seek technologies in which the return to specialization is less than exponential. It is convenient to work directly with the function $f(k)$, with the corresponding specialization coefficients given by:

$$\lambda_j = \frac{f(j+1)}{f(j)}.$$

To state our assumption on $f(k)$, notice that the reliability function $R(k)=(1-r)((1-p)/(1-r))^k$ is strictly increasing in k and for $R \leq 1-r$ can be inverted as

[1] Invention of pharmaceuticals and movies would seem to come the closest.

$$k = \frac{\log(1-r) - \log R}{\log(1-r) - \log(1-p)}.$$

This enables us to write expected output as a function of the reliability rate:

$$Y = F(R) = Rf\left(\frac{\log(1-r) - \log R}{\log(1-r) - \log(1-p)}\right).$$

Although in principle R is restricted to the grid induced by discrete values of k it will often be convenient to treat it as a continuous variable, which we will do without further comment.

Our basic assumption is

Assumption 1*:* $F(R)$ *is continuous, strictly concave and* $F(0) = 0$.

To better understand assumption 1, consider the family of technologies $f(k) = (b+k)^\alpha$ where $\alpha > 0$, $b \geq 0$ and $\beta = (b+1)^\alpha$. Notice that is an increasing function and that α measures the return to specialization. In addition it is apparent that $F(R)$ is continuous and $F(0) = 0$. We also have

Lemma 2: $F(R)$ *is strictly concave if and only if either* $\alpha \leq 1$ *or*

$$b \geq \frac{1-\alpha}{\log(1-r) - \log(1-p)}$$

Remark 1: In the case $\alpha \leq 1$ we have $f(k)$ concave, which we refer to as *diminishing returns to specialization*. When there are increasing returns to specialization, the gains to increasing the length of small chains may be so great that $F(R)$ fails to be concave. However, this is not the case if we choose b sufficiently large.

Proof: Define $\lambda = \log(1-r) - \log(1-p) > 0$. We may compute the second derivative

$$F_{RR}(R) = -\frac{\alpha}{\lambda}\left(b + \frac{\log(1-r) - \log R}{\lambda}\right)^{\alpha-1}\left(1 + \frac{1-\alpha}{\lambda}\left(b + \frac{\log(1-r) - \log R}{\lambda}\right)^{-1}\right)$$

Since the second factor is increasing in R, $F(R)$ will be strictly concave if and only if $F_{RR}(R)$ is non-negative at the upper bound $R = 1-r$, that is to say

$$1 + \frac{1-\alpha}{\lambda}b^{-1} \geq 0.$$

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Remark 2: Note that $R = 1 - p$ corresponds to a 1-chain $k = 1$. Since $F(0) = 0$ and $F(1-p) > 0$ there is some $\underline{R}$ such that for all smaller R we have $F(R) < F(1-p)$, that is, a 1-chain yields higher expected output – as well as greater reliability – than any chain of length greater than $k(\underline{R})$.

We can characterize the concavity of $F(R)$ by a bound on the derivatives of the primitive $f(k)$ with a Lemma proven in the Appendix.

Lemma 2: $F(R)$ *is strictly concave if and only if*

$$\frac{f_{kk}(k)}{f_k(k)} \leq \log(1-r) - \log(1-p).$$

In particular it is sufficient that $f(k)$ be concave.

For some results we will also need to insure that $f(k)$ not be too concave.

Definition 3: *We say that* $f(k)$ *is* moderately concave *if*

$$\frac{f_{kk}(k)}{f_k(k)} \geq \frac{f_k(k)}{f(k)} - k^{-1}$$

This is satisfied for the class $f(k) = (b+k)^{\alpha}$ with $b \geq 0, \alpha > 0$, in which case

$$f_{kk}(k)/f_k(k) = (f_k(k)/f(k)) - (b+k)^{-1} \geq (f_k(k)/f(k)) - k^{-1} \quad .$$

3. An Example

We start by studying the relationship between expected output and the length of a chain k in the simple case $f(k) = k^{\alpha}$. In this case expected output is

$$Y = R\left(\frac{\log(1-r) - \log R}{\log(1-r) - \log(1-p)}\right)^{\alpha}$$

As indicated, we will allow R to be a continuous variable, so we can maximize expected output using calculus. The derivative is

$$\begin{aligned} Y &= \left(\frac{\log(1-r) - \log R}{\log(1-r) - \log(1-p)}\right)^{\alpha} - \alpha\frac{(\log(1-r) - \log R)^{\alpha-1}}{(\log(1-r) - \log(1-p))^{\alpha}} \\ &= \frac{(\log(1-r) - \log R)^{\alpha-1}}{(\log(1-r) - \log(1-p))^{\alpha}}[\log(1-r) - \log R - \alpha] \end{aligned}$$

The sign of the derivative is determined by the term in square brackets, which is decreasing in R. This implies that $F(R)$ is single peaked[2] even for $\alpha > 1$ and has a maximum that is determined by a unique solution to the first order condition

$$\hat{R} = (1-r)e^{-\alpha}.$$

From this the optimal chain length is

$$\hat{k} = \frac{\alpha}{\log(1-r) - \log(1-p)}.$$

This simple formula shows how the optimal degree of specialization is determined by the trade off between the increased output of longer chains, and the increased failure rate. A greater return to chain length as measured by α means longer chains; higher probability of failure as measured by p means shorter chains; and higher correlation of shocks as measured by r means longer chains. Notice also that the optimal reliability rate $\hat{R}$ is independent of the failure rate p. It is decreasing in both α and r, so that as correlation increases, chain length increases so much that unemployment goes up.

We may summarize this as

Proposition 4: *Greater returns to specialization α, lower failure probability p, or higher correlation of shocks r and expected output maximization imply higher expected output Y, more specialization k and no less unemployment U. Greater returns to specialization and higher correlation of shocks strictly increase unemployment U, which is independent of the failure probability. Optimal chain length is*

$$\hat{k} = \frac{\alpha}{\log(1-r) - \log(1-p)}$$

and optimal unemployment is $\hat{U} = 1-(1-r)e^{-\alpha}$.

4. An Economy of Chains

We now consider a continuum economy. The shocks to chains and individuals are taken to be independent, which we interpret as meaning that a certain fraction of chains

[2] If k is restricted to the natural numbers, this means that the expected output maximizing choice of chain length must be at one of the grid points adjoining the point $\hat{k}$ that maximizes expected output over the positive real numbers

and individuals fail. That is, we assume that there is no aggregate uncertainty caused by the failure of individuals or chains. We index the production function by α, so that $y = f(k,\alpha)$.

The sole friction in the economy is an insurance market imperfection. Following Kehoe and Levine [1993], we assume that only a fraction γ of output can be used to make insurance payments. If $\gamma = 1$ we are in a frictionless full insurance world. If $\gamma = 0$ no insurance is possible. The economy has several stages:

- Complete contingent insurance markets
- Determination of chain length.
- Realization of an aggregate shock $(\alpha, p, r, \gamma) \in S$, where S is a finite set.
- Realization of individual shocks
- Output produced and insurance claims paid

This is a public information economy and our notion of equilibrium will be that of constrained efficiency – that is maximization of the welfare of the ex ante identical individuals in this economy. Note, however, that this is equivalent to a competitive equilibrium – that is, the first and second welfare theorems hold in this economy. To see this, observe that there are only finitely many relevant production technologies k: as we observed production technologies for which $k > \max_{(\alpha,p,r,\gamma)\in S} k(\underline{R}(\alpha,p,r))$ are dominated by 1-chains. Second, observe that the insurance market friction may be equivalently modeled by a physical production technology. That is, we may imagine that the input x jointly produces two kinds of output: $\gamma f(k)x$ units of tradable output, and $(1-\gamma)f(k)x$ of untradable output. This economy is completely a classical continuum general equilibrium economy, and so satisfies the welfare theorems – the second welfare theorem being trivial, as everyone is *ex ante* identical. Notice, however, that it may be efficient to put fractions of the population in different length chains; implicitly we are assuming that individual can be allocated to chains by means of lotteries, and that these lotteries are traded or that sunspot contingencies are available as in Kehoe, Levine and Prescott [2002].

5. No Aggregate Shock

We consider the case of an economy with no aggregate shock, so just a single aggregate state. We first show that it is not optimal to use lotteries or sunspot contingent chain lengths – there is an optimal chain length. Second, we examine when the comparative statics of output maximizing chain length in Proposition 1 extend to the general case. Throughout we assume that the length of chains are not restricted to integer values.

As remarked above we can have an economy consisting of chains of many different lengths, with individuals assigned to chains by lottery. Specifically, we can have a probability measure μ over a space Ω, with a measurable function $k(\omega)$ describing the corresponding length of chain. We may equally well describe the lottery by a measurable function $R(\omega)$ rather than $k(\omega)$.

If a chain at ω is successful it produces $F(R(\omega))/R(\omega)$. It consumes what is left over after all seizable output is seized plus an insurance payment $z_1(\omega)$, receiving in total $(z_1(\omega)+(1-\gamma)F(R(\omega))/R(\omega))$. If it is unsuccessful, it consumes just an insurance payment $z_0(\omega)$. Feasibility for insurance payments requires that $z_1(\omega), z_0(\omega) \geq 0$ and that the aggregate amount of insurance payments can be no greater than the seizable fraction of expected output

$$\int [R(\omega)z_1(\omega)+(1-R(\omega))z_0(\omega)]\mu(d\omega) \leq \gamma \int F(R(\omega))\mu(d\omega).$$

Proposition 5*: The optimal lottery* μ *is degenerate and places weight one on a single value of* k.

The proofs of the Propositions in this section can be found in the Appendix.

The utility function $u(c)$ induces preferences over expected output and reliability. We can round-out our picture of the economy be describing these preferences. Let Y be the expected per capita output of the common optimal chain. If the largest possible transfer is made from the successful to unsuccessful chains, then a successful chain gets $(1-\gamma)Y/R$, and a failed chain gets $(\gamma Y/R)(R/(1-R)) = \gamma Y/(1-R)$. If it is feasible for the failed chain to get more output than a successful chain $(1-\gamma)(1-R) \leq \gamma R$ then it is optimal to provide full insurance. Equivalently, we may write this as $R \geq 1-\gamma$. For these values of R only specialized output matters, and indifference curves are horizontal.

We now consider what happens when there is only partial insurance $R < 1-\gamma$. The *welfare function* – expected utility – is

$$W(R,Y) = Ru((1-\gamma)Y/R) + (1-R)u(\gamma Y/(1-R)).$$

We observe that $Ru((1-\gamma)Y/R)$ is homogeneous of degree one, and since u is concave it is also concave. The same is true of $(1-R)u(\gamma Y/(1-R))$. As W is the sum of these two functions, it is also concave.

Proposition 6*: The welfare function is concave in* R,Y *and strictly increasing in* Y*. Indifference curves are for* $R \leq 1-\gamma$ *smooth and downward sloping, and for* $R \geq 1-\gamma$ *horizontal. Indifference curves are differentiable including at* $R = 1-\gamma$.

Comparative Statics

In order to do comparative statics, we will need more assumptions about functional form. For the given utility function $u(c)$ we may define as usual the coefficient of relative risk aversion $\rho(c)$.

Definition 7: *We say that risk aversion is* moderate *if* $(\rho(c)/c)[1-\rho(c)] + \rho'(c) \geq 0$.

In the CRRA aversion case, this is true if and only if $\rho \leq 1$, that is, there is no more risk aversion than exhibited by the logarithm. In this case the utility function is $c^{1-\rho}$ and which we describe as a *moderate CRRA.*

Under moderate risk aversion we can establish the effect of better insurance

Proposition 8: *With moderate risk aversion, if there is partial insurance then increasing* γ *lowers reliability* R*, raises unemployment* $U = 1-R$ *and increases welfare, specialization and chain length* k.

What determines financial sector efficiency γ? This is a static model – to properly study insurance requires a dynamic setting since savings, portfolio balancing and borrowing along with bankruptcy and traditional unemployment insurance all form part of the overall insurance against unemployment. We can identify improved γ broadly with an improved financial sector. As we would expect, a better financial sector raises welfare. It does so by encouraging investment in riskier projects – that is to say, it encourages greater specialization by spreading the risk of failure. The striking fact is that by doing so it also raises the risk of failure: we should expect to find higher unemployment associated with better financial markets.

Bankruptcy may perhaps be overlooked as a form of insurance in this context – but one of the major causes of bankruptcy is job loss, and the ability to repudiate debt is an important form of insurance. You may recall that before the current crisis bankruptcy laws in the U.S. were tightened making it harder to go bankrupt. This lowers γ by making it more difficult to transfer resources from the employed – the lenders – to the unemployed, and of course lowers welfare. Why would anyone lobby for the government to take actions that lower γ? The thing to bear in mind is that when the law was changed there was already a great deal of outstanding debt that was made more difficult to discharge. Hence there was a one-time transfer from borrowers who might like to default to lenders. Clearly the credit companies – the lenders – took the view that their short-term gain more than offset the long-term loss caused by the fact that less debt – and insurance – would be acquired in the future. Hence the pursuit of a one-time transfer payment led to a decreased efficiency of the economy.

As our last result on the economy without shocks, we extend can extend the comparative statics of the example in Proposition 3. We parametrize the production function, considering $f(k,\alpha)$.

Definition 9: *We say that* α increases returns *to specialization if* $y_\alpha(k) > 0$ *and*

$$\frac{d(f_k(k,\alpha)/f(k,\alpha))}{d\alpha} > 0$$

This says output increases but that the marginal product of specialization is increased more.

Proposition 10*: Suppose that preferences are a moderate CRRA. (i) If* α *increases returns to specialization then higher* α *leads to higher expected output* Y *more specialization* k *and more unemployment* U*. (ii) Under moderate concavity of the production function a lower failure probability* p *or a higher correlation of shocks* r *leads to higher expected output* Y *more specialization* k *and no less unemployment* U*; higher correlation of shocks leads to strictly more unemployment.*

Optimal reliability under full insurance given by the unique solution $\hat{R}$ *to*

$$F_R(\hat{R}) = f\left(\frac{\log(1-r)-\log\hat{R}}{\log(1-r)-\log(1-p)}\right) - \frac{f_k\left(\frac{\log(1-r)-\log\hat{R}}{\log(1-r)-\log(1-p)}\right)}{[\log(1-r)-\log(1-p)]} = 0$$

If $\hat{R} \geq 1-\gamma$ *there is full insurance. Otherwise there is partial insurance and optimal reliability is determined as the unique solution to*

$$\left[\rho\hat{R}^{\rho-1}(1-\gamma)^{1-\rho} - \rho(1-\hat{R})^{\rho-1}\gamma^{1-\rho}\right] + (1-\rho)\left[\hat{R}^{\rho}(1-\gamma)^{1-\rho} + (1-\hat{R})^{\rho}\gamma^{1-\rho}\right](F_R(\hat{R})/F(\hat{R})) = 0$$

Notice that the optimal reliability is a strictly decreasing function of chain length, so the optimal chain length is given by

$$\hat{k} = \frac{\log(1-r)-\log\hat{R}}{\log(1-r)-\log(1-p)}$$

and optimal unemployment by $\hat{U} = 1-\hat{R}$.

One question is whether there is any empirical sense to the idea that poor countries have lower unemployment rates than rich countries. Using data from the CIA World Factbook, and eliminating tiny countries such as Monaco, and oil-producing countries such as Qatar, the countries with lowest unemployment are

Uzbekistan	1.1%
Thailand	1.5%
Cuba	1.7%
Papua New Guinea	1.8%
Bermuda	2.1%
Tajikistan	2.2%
Laos	2.5%
Mongolia	2.8%

which are not only poor countries, but include a large number of communist or formerly communist countries, a point we will return to. However, the countries with the highest unemployment are also very poor: Zimbabwe with 95% unemployment and Liberia with 85% unemployment being the extreme cases. However, in those cases unemployment seems to have less to do with specialization than with the absence of functioning political systems – which is to say, real unemployment and economic success is not explained only by long production chains, but the absence of civil wars and other civil disruptions also play a crucial role.

7. A Low Probability Negative Shock

We now turn to the impact of a shock. We examine the simplest case of a single negative shock. The crucial fact is that the shock hits after chain length has been determined. Our main goal is to establish how longer chains and lower correlation function as multipliers increasing the impact of a particular shock.

We will parametrize the production function, taking α as a parameter that increases output y. We consider the situation where there is a low probability of a negative shock. Specifically, we suppose that there is a baseline shock $(\alpha_0, 1-p_0, r_0, \gamma_0)$ with probability $1-\pi$ and a negative shock $(\alpha, 1-p, r, \gamma) \le (\alpha_0, 1-p_0, r_0, \gamma_0)$ with probability π. We suppose that the probability of the negative shock is sufficiently small that the optimal chain length $\hat{k}$ is approximately what it would be when $\pi = 0$.

Fixing the chain length k what is the impact of a negative shock on welfare, aggregate output, total factor productivity (TFP) and unemployment?

Production Function Shocks

A reduction in the benefits of the specialization technology α lowers welfare and aggregate output and has no effect on unemployment. Since the same amount of employment produces less aggregate output, TFP falls. Since the condition for full insurance is $R \ge 1-\gamma$ and R is not changed, with a fixed chain length k and a change only in α, the financial constraint will be binding if and only if it was binding in the base state. Notice that the impact of α on aggregate output is f_α. This is increasing in k if $f_{k\alpha} > 0$; in particular this is the case if α increases the returns to specialization.

Failure Rate Shocks

An increase in the failure rate p lowers welfare, lowers aggregate output and either raises unemployment or leaves it unchanged. It has no effect on TFP since the output per employed worker does not change. Since R declines, a financial constraint that does not bind in the base state may bind in the bad state further lowering welfare. Recall that reliability is given by

$$R = (1-r)\left(\frac{1-p}{1-r}\right)^k$$

from which we may find the quantitative impact of a shock to p

$$R_p = k\left(\frac{1}{1-r}\right)^{k-1}(1-p)^{k-1}.$$

The longer is the chain k and the higher is the correlation r the more sensitive is reliability and unemployment to changes in p. This is a kind of "reverse Keynesian multiplier" where the impact of a shock is greatly increased by the fragility of the chains that make up the economy. Notice that there is kind of a double-impact of the base level of correlation. If base correlation is high then this implies that the optimal choice of k is large, so that R_p will be very large indeed.

Since the production function f and chain length k do not change in response to a failure rate shock, as we observed, TFP does not change. That is, decreased output is due entirely to decreased hours worked. It is less apparent that *measured* TFP will be unchanged. Specifically, in the model everyone working for an unproductive chain is counted as "unemployed." In practice this is neither the definition of unemployment used in collecting statistics, nor practically what it measures. That is, one specialized firm in a failed chain may close down laying off workers who become "unemployed" while another specialized firm in the same or a different failed chain may simply work hard while failing to produce any useful output. The economic consequences are the same in both cases, yet in one case the firm contributes unemployment and a reduction of hours worked, while in the other the firm contributes no unemployment or reduction in hours worked.

Specifically, suppose that unemployment U in the model translates as a fraction of workers who are measured as unemployed ηU and a fraction who are measured as employed but who are in fact unproductive $(1-\eta)U$. In this case an increase in the underlying variable U raises unemployment, but it also reduces the hours worked less than the reduction in output, that, TFP is output divided by hours worked

$$\frac{(1-U)f(k)}{(1-\eta U)} < f(k)$$

and so measured TFP falls, although actual TFP does not. This should be a warning against paying too much attention to measures such as unemployment or TFP: measured employment and hours worked do not distinguish between individuals who play cards at home because they have been laid off, and those who play cards at work because their production line has been shut down by a shortage of parts. From the overall viewpoint of the economy, however, both are equally unproductive.

Correlation Shocks

A decrease in the correlation r lowers welfare, lowers aggregate output and raises unemployment. Like a shock to the failure rate, it has no effect on TFP although it may reduce measured TFP. Since R declines, a financial constraint that does not bind in the base state may bind in the bad state further lowering welfare. The quantitative impact of a shock to r is

$$R_r = (k-1)\left(\frac{1}{1-r}\right)^{k-2}(1-p)^k .$$

which exhibits the same kind of sensitivity to long chains and a high underlying correlation rate as does shocks to the failure rate.

A key issue is the correlation of p and r. That is, when we measure the size of a shock, we are likely to measure the change in p. Consider, for example, a shock to a resource prices. Suppose we measure the size of shock by the increase in the cost to economy of continuing the same level of resource usage as in the base state. This is a measure of p: the fraction of the economy that will fail due to these increased costs we imagine is roughly proportion to the increase in cost. However, the overall impact of the shock is measured by the number of chains that fail, not the number of individuals that fail, and this depends on the correlation. Different shocks of the same size p may have different correlations r. For example, crude oil shocks may have a very low r because the failures they cause are spread across chains, while linseed oil shocks may have a very high r because they fall primarily in a single chain, and there are easy substitutes. Hence a shock to the price of crude oil and a shock to the price of linseed oil that have the same overall economic cost as measured by change in price times sales will have very different economic consequences.

Dynamic Shocks

Next, consider the temporal dimension of the shock. The shock here is assumed to take place after chain length is determined. When we studied the comparative statics of chains without a shock, implicitly we were examining what happens to chain length that is determined after a shock. To make this formal, consider three periods. There are no temporal connections between the periods – no savings or investment. However we allow for a correlation structure in the shocks. In particular, if last period was the base state, the probability of the bad state in the current period is small: π. However, if last period was the bad state, then in the current period the probability of the base state is also π. That is, we assume a high degree of serial correlation in the states. The initial condition is the base shock.

Consider a shock to either p or r or both. In the first base period output and TFP are high and unemployment is low. In the second period unexpectedly a shock hits. This lowers output and possibly TFP and raises unemployment. In the third period it is now very likely that the negative state will remain. This implies a lower optimal chain length, further lowering output and definitely lowering TFP. However: it also lowers unemployment – below even what it was in the base state as we know from Proposition 10. In other words, output and TFP fall continually, while unemployment spikes up then drops back down.

Financial Shocks

A reduction in the efficiency of the financial sector γ can lower welfare, but has no effect on measurable variables such as aggregate output, TFP or unemployment. In order to have an impact, the financial constraint must bind in the bad state. Notice that if it binds in the good state, it necessarily binds in the bad state.

An important consideration is whether γ is accurately known. Take the case where in the base economy the insurance constraint is not binding – there is full insurance. The value of γ will only matter when the negative shock hits; this is infrequent so wrong beliefs will be slow to be corrected. This can be viewed as a kind of "quasi" self-confirming equilibrium. Self-confirming equilibrium allows wrong beliefs about events that are not observed. However, wrong beliefs about events that are rare can persist for a long period of time, so are not that different from events that are not observed at all. Moreover, the financial sector – which profits by selling more "insurance" has a strong incentive to exaggerate the efficacy of that insurance – as was in fact done in recent years by issuing numerous loan guarantees not backed by any resources that could be used to honor the guarantees. Hence the promised γ may be larger than the actual γ. This will only be discovered after the negative shock hits, at which point there will be a "crisis" in the sense of the existence of many promises impossible of fulfillment. At which point there will be a fight over who gets shortchanged, and Lloyd Blankfein gets on the phone to Hank Paulson and says "It shouldn't be us – grab 800 billion from the Treasury."

The key point is: regardless of who winds up bearing the burden of unfulfillable promises there will be a strong temptation to say "This should never happen again" and tighten up financial regulation – and by doing so reduce the value of γ. This of course will reduce welfare, and in the future when chain length can adjust, lead to a long-term reduction in output and welfare. Japan and the United States seem to have been especially proficient at responding to crises by strangling their financial sectors: Both countries have allowed large failed banks to pretend to be solvent by holding treasury securities rather than lending to the private sector. By way of contrast Chile and Sweden responded to crises by explicitly failing their banks, and creating new banks that could carry on a normal banking business.

Short Chains

Efficient economies with large values of γ, α, r and $1-p$ will have long chains. As we observed, long chains are vulnerable chains, and output and welfare will decline more in response to shocks – possibly substantially more. This can of course be avoided. For example, the capitalist economy can be replaced by a government planned economy, and the government may (and probably will) choose the length of chain k lower than the optimal level $\hat{k}$. This reduces welfare, but also reduces volatility. It also reduces unemployment, both in the base state and following a shock. All of which suggests that measures to reduce unemployment should be viewed with a degree of skepticism. If the model of production chains is correct, then high volatility is the price we must pay for high welfare.

Notice also that choosing short chains while it may reduce unemployment following a shock, does not necessarily increase welfare even *ex post* following the negative shock. Take the case of a chain of length 1. This exhibits the least volatility, but also the least aggregate output in any state. If γ is high, there can be full insurance even in the bad state: in this case it is better to be unemployed in the bad state at $\hat{k}$ than to be employed in the "socialist paradise". Indeed, we may well ask: is it better to be employed in Cuba or unemployed in the United States? In fact weekly pay in Cuba is about $187,[3] while unemployment benefits in the United States are about $300.[4]

This analysis suggests a natural experiment. From the end of the Second World War until it entered the oil market in a large way in the 1970s, the Soviet bloc was a closed and planned economy. Did it have shorter production chains than the West? Did it have lower unemployment and less volatility? Can this be accounted for by the different length of production chains? The first step in applying this type of theory is to develop some effective measure both of the length of chains, and the correlation r. Unfortunately there may not be reliable data on the old Soviet block – economic statistics from that period consist mostly of lies, but insofar as reliable data can be found, it seems a good place to look. Even the very crude examination of unemployment statistics above shows the lowest unemployment rates dominated by communist and former communist countries.

International Trade Considerations

We have considered a closed economy. What about trade between countries? Two considerations come to mind. First, chains can overlap countries. There may be many chains in a large country, only a few of which extend to a smaller country. Shocks that impact the sector that overlaps will have a large impact on the small country, but only a small impact on the large country: when the United States coughs, Mexico catches cold.

The other application is Foreign Direct Investment (FDI). FDI has a disproportionately beneficial effect on developing countries. These small countries with

[3] The CIA World Factbook reports annual per capital income in Cuba of 9700 PPP adjusted 2009 U.S. dollars.
[4] U.S. Department of Labor.

poorly developed financial sectors and few ties to outside suppliers will find it optimal to choose short production chains. A large multinational can create a chain that lies mostly outside the country, and has access and connections to supplier worldwide. Hence it has an incentive to create longer and more productive chains. After a time, knowledge of outside suppliers and business connections will spread to other producers in the developing country – effectively increasing their r and making it efficient for domestic producers to create longer and more productive chains. It is widely understood that the spread of knowledge is the likely suspect for the disproportionate effect of FDI. The model of production chains suggest a possible mechanism through which this acts.

8. Conclusion

At the heart of this paper is the importance of understanding whether fluctuations are efficient or not. Are they the price we pay for wealth? If so, far from being a problem, they are a solution – and policies to mitigate them can be counterproductive. This can be seen by the possibility of creating inefficiently short changes in order to mitigate volatility.

The model of chains also helps understand the consequences of technological change. For example, better communications – the internet – can potentially increase r. This will lead to greater specialization, longer chains, more output, and higher welfare. But it will also result in higher unemployment and greater volatility.

In short – the capitalist economy may indeed be more prone to crisis – and this may be a good thing.

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Appendix

Define the constant $\lambda = \log(1-r) - \log(1-p) > 0$.

Lemma 2: $F(R)$ *is strictly concave if and only if*

$$\frac{f_{kk}(k)}{f_k(k)} \leq \log(1-r) - \log(1-p).$$

Proof: We compute the derivatives of $F(R)$:

$$F = Rf\left(\frac{\log(1-r) - \log R}{\lambda}\right)$$

$$F_R = f\left(\frac{\log(1-r) - \log R}{\lambda}\right) - \lambda^{-1} f_k\left(\frac{\log(1-r) - \log R}{\lambda}\right)$$

$$F_{RR} = -(\lambda R)^{-1} f_k + \lambda^{-2} R^{-1} f_{kk}$$

Hence, $F_{RR}(R) \leq 0$ if and only if $-\lambda f_k + f_{kk} \leq 0$, which is the desired condition.

☑

Proposition 5*: The optimal lottery* μ *is degenerate and places weight one on a single value of* k.

Proof: Consider

$$\overline{R}(\omega) = \overline{R} \equiv \int R(\omega)\mu(d\omega).$$

and the insurance scheme $\overline{z}_0(\omega) = z_0(\omega)$,

$$\overline{z}_1(\omega) = \overline{z}_1 \equiv \int R(\omega) z_1(\omega)\mu(d\omega) / \overline{R} \geq 0.$$

We will show that this scheme does no worse than the original one, and if the original one is non-degenerate, strictly better.

Since $Y(R)$ is strictly concave

$$\begin{aligned} &\overline{R}\overline{z}_1 + \int [(1 - R(\omega))\overline{z}_0(\omega)]\mu(d\omega) \\ &= \int [R(\omega)z_1(\omega) + (1 - R(\omega))z_0(\omega)]\mu(d\omega) \\ &\leq \gamma \int F(R(\omega))\mu(d\omega) < \gamma F(\overline{R}) \end{aligned}$$

so the proposed scheme satisfies the insurance feasibility constraint. Notice that under the new scheme, there are exactly as many unemployed as under the old scheme, and that they get exactly the same insurance payments. The expected payment to an employed person under the old scheme is

$$\begin{aligned}\overline{c}_1 &= \int R(\omega)\left(z_1(\omega) + (1-\gamma)Y(R(\omega))/R(\omega)\right)\mu(d\omega)/\overline{R} \\ &= \overline{z}_1 + (1-\gamma)\int Y(R(\omega))\mu(d\omega) \\ &< \overline{z}_1 + (1-\gamma)Y(\overline{R})\end{aligned}$$

where the inequality follows because $Y(R)$ is strictly concave, and is strict if the original lottery is non-degenerate. Under the old scheme the employed received utility.

$$\int R(\omega)u\left(z_1(\omega) + (1-\gamma)P(R(\omega))/R(\omega)\right)\mu(d\omega)/\overline{R} \le u(\overline{c}_1) < u(\overline{z}_1 + P(\overline{R}))$$

where the first inequality is Jensen's inequality. This shows the new scheme is better.

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Proposition 6*: The welfare function is concave in* R, Y *and strictly increasing in* Y*. Indifference curves are for* $R \le 1-\gamma$ *smooth and downward sloping, and for* $R \ge 1-\gamma$ *horizontal. Indifference curves are differentiable including at* $R = 1-\gamma$*.*

Proof: We already observed in the text that $W(R, Y)$ is concave. To do the relevant computations, set

$$\begin{aligned}a_0 &= \gamma/(1-R) \le 1 \\ a_1 &= (1-\gamma)/R \ge 1\end{aligned}$$

with exact equality if and only if $R = 1-\gamma$. We then compute the partial derivatives of the welfare function $W(R,Y) = Ru\left((1-\gamma)Y/R\right) + (1-R)u\left(\gamma Y/(1-R)\right)$

$$\begin{aligned}W_R &= u((1-\gamma)Y/R) - u(\gamma Y/(1-R)) \\ &\quad + \frac{\gamma u'(\gamma Y/(1-R))Y}{1-R} - \frac{(1-\gamma)u'((1-\gamma)Y/R)Y}{R} \\ &= u(a_1 Y) - u(a_0 Y) + a_0 Y u'(a_0 Y) - a_1 Y u'(a_1) \\ W_Y &= (1-\gamma)u'((1-\gamma)Y/R) + \gamma u'(\gamma Y/(1-R))\end{aligned}$$

We observe that W_R vanishes at the full insurance line $R = 1-\gamma$, and is negative to the left since u is concave.

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Proposition 8: *With moderate risk aversion, if there is partial insurance then increasing γ lowers reliability R, raises unemployment $U = 1 - R$ and increases specialization and chain length k.*

Proof: Notice that changes in γ do not change $F(R)$. In the proof of Proposition 6 we computed the marginal utilities W_R, W_Y. Differentiating each with respect to γ we find

$$\begin{aligned} W_{R\gamma} &= -Yu'(a_1 Y)/R - Yu'(a_0 Y)/(1-R) + Yu'(a_0 Y)/(1-R) + Yu'(a_1)/R \\ &\quad + a_0 Y^2 u''(a_0 Y)/(1-R) + a_1 Y^2 u''(a_1)/R \\ &= a_0 Y^2 u''(a_0 Y)/(1-R) + a_1 Y^2 u''(a_1)/R < 0 \end{aligned}$$

$$W_{Y\gamma} = u'(a_0 Y) - u'(a_1 Y) + a_0 Y u''(a_0 Y) - a_1 Y u''(a_1 Y)$$

A sufficient condition for $W_{Y\gamma} > 0$ is that $u'(c) + cu''(c)$ be decreasing in c. Observe that

$$\begin{aligned} D\rho &= D\frac{cu''}{u'} = \frac{-u''u' - cu'''u' + c[u'']^2}{[u']^2} \\ &= -\frac{2u'' + cu''' - u''[1-\rho]}{u'} = -\frac{D(u' + cu'') - u''[1-\rho]}{u'} \end{aligned}$$

Rearranging we find

$$D(u' + cu'') = -u'[(\rho/c)[1-\rho] + \rho']$$

which is negative from the condition of moderate risk aversion.

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Proposition 10: *Suppose that preferences are a moderate CRRA. (i) If α increases returns to specialization then higher α leads to higher expected output Y more specialization k and more unemployment U. (ii) Under moderate concavity a lower failure probability p or a higher correlation of shocks r leads to higher expected output Y more specialization k and no less unemployment U; higher correlation of shocks leads to strictly more unemployment.*

Optimal reliability under full insurance given by the unique solution $\hat{R}$ to

$$F_R(\hat{R}) = f\left(\frac{\log(1-r) - \log \hat{R}}{\log(1-r) - \log(1-p)}\right) - \frac{f_k\left(\dfrac{\log(1-r) - \log \hat{R}}{\log(1-r) - \log(1-p)}\right)}{[\log(1-r) - \log(1-p)]} = 0$$

If $\hat{R} \geq 1-\gamma$ *there is full insurance. Otherwise there is partial insurance and optimal reliability is determined as the unique solution to*

$$\left[\rho\hat{R}^{\rho-1}(1-\gamma)^{1-\rho} - \rho(1-\hat{R})^{\rho-1}\gamma^{1-\rho}\right] + (1-\rho)\left[\hat{R}^{\rho}(1-\gamma)^{1-\rho} + (1-\hat{R})^{\rho}\gamma^{1-\rho}\right](F_R(\hat{R})/F(\hat{R})) = 0$$

Proof: In the case of full insurance, expected output must be maximized, and $F(R)$ is concave by assumption, so $F_R(R) = 0$. The value of $F_R(R)$ was computed in the proof of Lemma 2. Notice that $F_R(R)/F(R)$ has the same sign as $F_R(R)$, so to prove decreased reliability (or increased unemployment) it suffices to show that $F_R(R)/F(R)$ decreases at $F_R(R) = 0$.

Now consider $(1-r)e^{-\alpha} < 1-\gamma$ and partial insurance. We may substitute $F(R)$ into the utility function to find *ex ante* welfare as a function solely of reliability

$$\begin{aligned} w(R) &= R^{\rho}\left((1-\gamma)F(R)\right)^{1-\rho} + (1-R)^{\rho}\left(\gamma F(R)\right)^{1-\rho} \\ &= \left[R^{\rho}(1-\gamma)^{1-\rho} + (1-R)^{\rho}\gamma^{1-\rho}\right]F(R)^{1-\rho} \end{aligned}$$

and we know that this function is concave. Computing the derivative

$$\begin{aligned} w_R(R) = &\left[\rho R^{\rho-1}(1-\gamma)^{1-\rho} - \rho(1-R)^{\rho-1}\gamma^{1-\rho}\right]F(R)^{1-\rho} \\ &+(1-\rho)\left[R^{\rho}(1-\gamma)^{1-\rho} + (1-R)^{\rho}\gamma^{1-\rho}\right]F(R)^{-\rho}F_R(R) \end{aligned}$$

we then know that the unique optimum has $w_R(\hat{R}) = 0$ and that $w_{RR}(R) < 0$. So to sign changes in $\hat{R}$, we need only determine whether $w_R(\hat{R})$ increases or decreases: the optimal $\hat{R}$ must move in the same direction. Equivalently we may divide through by $F(R)^{1-\rho}$ since this is positive. This gives

$$\begin{aligned} w_R(R)/F(R)^{1-\rho} = &\left[\rho R^{\rho-1}(1-\gamma)^{1-\rho} - \rho(1-R)^{\rho-1}\gamma^{1-\rho}\right] \\ &+(1-\rho)\left[R^{\rho}(1-\gamma)^{1-\rho} + (1-R)^{\rho}\gamma^{1-\rho}\right]F_R(R)/F(R) \end{aligned}$$

This moves in the same direction as $F_R(R)/F(R)$, so as in the case of full insurance, it suffices to determine the direction of change of $F_R(R)/F(R)$.

Recall that

$$F(R) = Rf\left(\frac{\log(1-r) - \log R}{\lambda}\right),$$

so

$$F_R(R) = f\left(\frac{\log(1-r)-\log R}{\lambda}\right) - (1/\lambda) f_k\left(\frac{\log(1-r)-\log R}{\lambda}\right)$$

and

$$\frac{F_R(R)}{F(R)} = \frac{1}{R} - \frac{1}{\lambda}\frac{f_k\left(\frac{\log(1-r)-\log R}{\lambda}\right)}{f\left(\frac{\log(1-r)-\log R}{\lambda}\right)}$$

The direction of movement of reliability is determined by the direction of movement of $F_R(R)/F(R)$, and only the second term matters. In the case where α increases the returns to specialization, the second term decreases by definition, so specialization goes down. This forces a higher specialization rate k, and since the production possibility frontier shifted up, higher expected output.

Next observe that the direction of movement of $F_R(R)/F(R)$ is the same as that of

$$\begin{aligned} L(R) &\equiv -\log((1/R) - (F_R(R)/F(R))) \\ &= \log\lambda - \log f_k\left(\frac{\log(1-r)-\log R}{\lambda}\right) + \log f\left(\frac{\log(1-r)-\log R}{\lambda}\right) \end{aligned}$$

Increasing p changes only λ and makes it larger. Differentiating with respect to λ we get

$$L_\lambda = \lambda^{-1} + \lambda^{-1} k f_{kk}/f_k - \lambda^{-1} k f_k / f$$

which is non-negative by the assumption of moderate concavity of $f(k)$, so reliability goes up. This is possible with higher p only if specialization decreases. Also since the production possibility frontier moved down, higher reliability also implies less expected output.

Differentiating with respect to r and noticing that $d\lambda/dr = -(1/(1-r))$ we find

$$\begin{aligned} &L_r \\ &= -\frac{1}{\lambda(1-r)} - \frac{1}{\lambda(1-r)} k f_{kk}/f_k + \frac{1}{\lambda(1-r)} k f_k / f + \frac{1}{\lambda(1-r)} f_{kk}/f_k - \frac{1}{\lambda(1-r)} f_k / f. \\ &= -\frac{1}{\lambda(1-r)} - \frac{1}{\lambda(1-r)}((k-1) f_{kk}/f_k - (k-1) f_k / y) \end{aligned}$$

This has the same sign as

$$-(k-1)^{-1} - f_{kk} \,/\, f_k + f_k \,/\, y \,.$$

Moderate concavity says $0 \le k^{-1} + f_{kk} \,/\, f_k - f_k \,/\, f$, and since $(k-1)^{-1} > k^{-1}$, we conclude that L_r is negative. Hence reliability falls, this requires more specialization and also, since the production possibility frontier shifted up, expected output goes up.

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