Fairness, Risk Preferences and Independence: Impossibility Theorems

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Abstract: The most widely used economic models of social preferences are specified only for certain outcomes. There are two obvious methods of extending them to lotteries. If we do so by expected utility theory, so that the independence axiom is satisfied, our results imply that the resulting preferences do not exhibit *ex ante* fairness. If we do so by replacing certain outcomes with their expected utilities for each individual, so that individual risk preferences are preserved, then *ex ante* fairness may be preserved, but neither the independence axiom nor *ex post* fairness is satisfied.

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1. Introduction

If people have both a preference for fairness and continuous preferences, they are willing to make a small sacrifice for a more egalitarian outcome. One way to create "fairness" in an *ex ante* sense is to flip a coin and reverse the roles. The use of lotteries to allocate indivisible rewards and costs (such as the draft lottery) is evidence that *ex ante* fairness is often a concern. There is also considerable experimental evidence that agents care about *ex ante* fairness, see for example, Bolton, Brandts and Ockenfels [2005], Krawcyck and le Lec [2006], Bolton and Ochenfels [2009] and Kircher *et al* [2009]. We show that there is a conflict between *ex ante* fairness and the independence axiom. In particular, we point out that five leading theories of outcome-based social preferences for fairness, those of Fehr and Schmidt [1999], Bolton and Ockenfelds [2000], Charness and Rabin [2002], Cox and Sadirij [2004], and Andreoni and Miller [2002] all fail to reflect concerns for *ex ante* fairness because they are expected utility theories.[1]

One contribution of this paper is the formulation of very weak notions of *ex ante* fairness that are clearly inconsistent with the independence axiom. We conclude that no theory that incorporates a reasonable notion of *ex ante* fairness can maintain the assumption of expected utility. Our results are a formalization and generalization of examples and results from the social choice literature, starting from the Diamond [1967]/ Machina [1989] example of a parent or social planner who strictly prefers to use a coin flip to allocate an indivisible good to two (other) agents and thus violates the axioms of Harsanyi [1955]. The social choice literature, which adopts Harsanyi's "impartial-observer" viewpoint, has responded to this example by relaxing independence in various ways.[2] The conflict between independence and *ex ante* fairness has also been noted in the behavioral literature, for example Kircher et al [2009] give a verbal argument indicating that *ex ante* fairness is inconsistent with utilitarianism, and implicitly with the independence axiom.

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[1] This note studies only such outcome-based or "consequentialist" theories, as opposed to those that allow payoffs to depend directly on the beliefs and intentions of others, as in Rabin [1993].

[2] For example Karni and Safra [2002] make use of only a partial version of independence in their study of the use of lotteries to solve indivisibility problems. Grant et al [2010] also recognize this when they reconcile the Diamond paradox with Harsanyi's social choice theory: they do so by means of a social

To formalize the insight in the various past examples and experiments, we give formal and fairly weak definitions of fairness, using only the domain of "coin flip" lotteries. We consider both "fairness for you" – a willingness to sacrifice my payoff for your benefit – and "fairness for me" – a willingness to reduce your utility to achieve higher utility for me. Our first point is that if either of these holds in a very weak form *ex ante* then the independence axiom must be violated. As a consequence, if social preferences over deterministic allocations are extended to lotteries by treating the associated utility functions as expected utility, *ex ante* fairness is violated both for you and for me.

We also show that transitive, state-independent preferences that are consistent with a well defined notion of individual risk preference over own consumption necessarily violate a weak form of *ex post* fairness. This is true, for example, for Fehr-Schmidt preferences extended to lotteries by replacing money income with expected money income. More generally, preferences that depend only on the expected value of individual utilities, such as those in Grant et al [2010], may be *ex ante* fair, but cannot be *ex post* fair. More generally, none of the standard models of fairness and social preference used in experimental research and their obvious extensions to lotteries can incorporate both *ex ante* and *ex post* fairness. There are however relatively straightforward variants of these models that do reflect both concerns. We conclude that experimental research on social preferences should pay more attention to preferences over lotteries, and that decision theorists should then consider what classes of tractable preferences are broadly consistent with the resulting data.

2. Ex Ante Fairness for You and For Me

There are two players who we refer to as "me" and "you." We consider *certain outcomes* $(m,y) \in \Re_+^2$ that can be interpreted as money for me and money for you. We also consider simple lotteries generated by tossing a fair coin with equal 50% probability of H(eads) or T(ails). We call this a *coin flip* and it can be written as $((m^H, y^H), (m^T, y^T)) \in \Re_+^2 \times \Re_+^2$.

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welfare function that is a non-linear function of the expected utility of different individuals, and so violates the independence axiom

We are interested in "my" preferences over certain outcomes and coin flips given as a complete order $\succsim$ over $\Re_+^2 \cup \Re_+^4$, with the derived strict ordering $\succ$ and indifference relation $\sim$.[3]

Axiom 1 [Independence]: *If* $(m_1, y_1) \succsim (m_2, y_2)$ *then for any* (m^H, y^H) *we have* $((m^H, y^H), (m_1, y_1)) \succsim ((m^H, y^H), (m_2, y_2))$.

Notice that since the role of the two certain prospects may be reversed, the same holds for the indifference relationship. Any expected utility theory – that is preferences induced by a utility function linear in probabilities – must satisfy Axiom 1, and some non-expected utility theories satisfy the axiom as well. It is weaker than the usual independence axiom, in the sense that it needs to hold only for fair coin flips; in particular since prospect theory models do not distort the probabilities $\{0, 1/2, 1\}$ they satisfy this axiom.[4] On the other hand at an intuitive level it is clear that the axiom conflicts with the idea that people might prefer lotteries that are ex ante fair.

The most immediate conflict between the independence axiom and fairness comes from an *ex ante* version of fairness, so we start from that. Below we show there is also a contradiction between independence and *ex post* notions of fairness under the standard continuity assumption.

To motivate our definition of *ex ante* fairness, suppose the agent weakly prefers $(8,5)$ to the more egalitarian $(7,7)$, and strictly prefer the fairer coin-flip lottery $((1000,0),(7,7))$ to the less fair $((1000,0),(8,5))$. Then the next Axiom is satisfied, regardless of preferences over other lotteries.

Axiom 2 [*Ex Ante* Fairness for You]: *There is a* $y_2 > y_1, m_2 > m_1$[5] *so that* $(m_1, y_2) \precsim (m_2, y_1)$ *and an* $m^H > m_2, y^H < y_1$ *such that* $((m^H, y^H), (m_1, y_2)) \succ ((m^H, y^H), (m_2, y_1))$.

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[3] It seems natural to assume that the order over certain outcomes is consistent with the order over coin flips, but we do not assume this as it is not required for our results.

[4] Herstein and Milnor [1953] derive the usual independence axiom from this axiom and a continuity assumption. If we replace weak preference with strict preference, this is what Machina [1989] refers to as "mixture separability."

[5] In this and the next axiom we use larger subscripts as a reminder of larger values.

This says that there is a sacrifice to make you better off that I would not make when comparing two deterministic outcomes, but I would make the sacrifice in the context of a coin flip if the other outcome is sufficiently less fair for you.

Remark: Suppose $m_2 = 1, y_1 = 4, m^H = 2, y^H = 3$. Then the outcome $(2,3)$ may seem more fair than $(1,4)$. However it might be "less fair" for you, even though it is farther from an equal division. Equal division *is* identified with fairness in many economic models, but in general depends on the units in which things are measured, and one can imagine situations where I think it is fair that you get four times as much as I do. Here we take the approach of being agnostic about what division is "fair." However, we could add to Axiom 1 the restriction that $m_2 > y_1$ (to reflecting the notion that fairness means equal division) without any substantive change in our results.

Theorem 1: *There are no preferences satisfying Axioms 1 and 2.*

Proof: Axiom 1 asserts that there can be no parameters such that the conditions in Axiom 2 hold; while Axiom 2 asserts the existence of some parameters, so the contradiction is immediate.

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In particular Example 1 violates the independence axiom.

Ex ante fairness for me is just the flip side of *ex ante* fairness for you.

Axiom 3 [*Ex Ante* Fairness for Me]*: There is a* $y_2 > y_1, m_2 > m_1$ *so that* $(m_1, y_2) \succsim (m_2, y_1)$ *and an* $m^H < m_1, y^H > y_2$ *such that* $((m^H, y^T), (m_2, y_1)) \succ ((m^H, y^T), (m_1, y_2))$.

This says that there is an opportunity to enrich myself at your expense I would not take, but I am concerned about *ex ante* fairness for me in the sense that I would exploit it in the context of a coin flip if the other outcome is "sufficiently less fair" for me.

For the same reason as Axiom 2, Axiom 3 is a transparent violation of independence.

Theorem 2: *There are no preferences satisfying Axioms 1 and 3.*

3. Economic Models of Social Preference

We now discuss four different social preferences from the literature that reflect a concern for fairness.[6] Each of them describes choices under certainty. One method of extending them to uncertainty is to treat them as expected utility functions, and evaluate lotteries by their expected value; in this case Axioms 2 and 3 must be violated. As we will see, alternative extensions to uncertainty, such as replacing lotteries over income with the expected value of income, run in to different problems.

Fehr and Schmidt

The Fehr and Schmidt [1999] social preferences, in our notation, are given by $U(m,y) = m - \alpha \max\{y-m,0\} - \beta \max\{m-y,0\}$ with $0 \le \beta < \alpha$. That is, if you are getting more than me I dislike it, and if I am getting more than you, I also dislike that, although not as much as I dislike you getting more than me. Although Fehr and Schmidt do not explicitly say this, it is implicit from their discussion and analysis that this is an expected utility function, so the independence axiom (axiom 1) is satisfied. Thus this version of their preferences violates Axiom 2 – *ex ante* fairness for you – as well as Axiom 3 – *ex ante* fairness for me.

Now consider the alternative of extending the preferences by replacing income with expected income. We will show that if $\alpha > \beta \ge 0$ that both Axioms 2 and 3 are satisfied, while of course the independence axiom is not. Let $m_1 = \alpha, y_2 = \alpha + 1$, and for any γ let $m_2 = \gamma\beta, y_1 = \gamma(\beta - 1)$, so that $(m_1, y_2) \sim (m_2, y_1)$ since both yield zero utility. Define $c(x) = \alpha \max(x,0) + \beta \min(x,0)$. Then

$$\begin{aligned}
\Delta(m^H, y^H) &\equiv 2U((m^H, y^H),(m_1,y_2)) - 2U((m^H, y^H),(m_2,y_1)) \\
&= m^H + m_1 - c(m^H + m_1 - y^H - y_2) \\
&\quad -(m^H + m_1 - c(m^H + m_2 - (y^H + y_2))) \\
&= \alpha - \gamma\beta + c(y^H - m^H - \gamma) - c(y^H - m^H + 1)
\end{aligned}$$

If $y^H - m^H < -1$, with $m^H > m_2, y^H < y_1$

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[6] Cox, Friedman and Sadiraj [2008] is somewhat related but considers axioms for "being more altruistic than" as opposed to preferences for fairness and does not consider the role of lotteries.

then

$$\begin{aligned}\Delta &= \beta(y^H - m^H - \gamma) - \beta(y^H - m^H + 1) + \alpha - \gamma\beta \\ &= \alpha - \beta - 2\gamma\beta\end{aligned}$$

For $\alpha > \beta$ we can choose $\gamma < (\alpha - \beta)/2\beta$ so that $\Delta > 0$, so Axiom 2 is satisfied. Notice if $\beta = 0$ we have $\Delta > 0$ for any γ. Intuitively, since I care about fairness for me a coin flip that is unfair to you lets me ignore the fact that in the alternative outcome you get more than me. Notice that $m_2 > y_1$ and $m_1 < y_2$ so that even if we added these constraints to the Axiom, it would still be satisfied by these Fehr-Schmidt preferences.

To show that Axiom 3 is satisfied let $y^H - m^H$ be greater than β, with $m^H < m_1, y^H > y_2$. Then

$$\begin{aligned}\Delta &= \alpha(y^H - m^H - \gamma) - \alpha(y^H - m^H + 1) + \alpha - \gamma\beta \\ &= -\gamma(\alpha + \beta)\end{aligned}$$

so $\Delta < 0$ and Axiom 3 is satisfied.

Notice that the results concerning Axiom 2 and 3 are not symmetric, and in particular Axiom 3 holds even if $\beta > \alpha$. The reason for this is that the Fehr and Schmidt utility function does not treat you and me symmetrically: the unfairness costs $c(x)$ are symmetric when $\beta = \alpha$ but the utility function contains my money income and not yours. This creates a bias in my favor, so that, in particular "fairness to me" may satisfied even as I am unconcerned about "fairness to you".

Bolton and Ockenfels

The Bolton and Ockenfels [2009] preferences are given by $U(m,y) = v(m, m/(m+y))$ where v is second differentiable and it is increasing and concave in the first argument, and concave with a maximum at ½ in the second argument. Here my utility depends on my share of the total as well as the amount of money I receive. Notice that is closely related to a variation of Fehr and Schmidt preferences in which differences are measured relative to the total[7]

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[7] It is not clear that measuring money relative to the total is desirable since it is not clear what the base amounts are supposed to be. For example if the amounts are receipts in a laboratory experiment, they may sum to zero or even negative.

$$U(m,y) = m - \alpha \max\{(y-m)/(y+m),0\} - \beta \max\{(m-y)/(y+m),0\}.$$

This differs from Bolton and Ockenfels only in that U is not differentiable. As is the case with Fehr and Schmidt if we assume the independence axiom to extend these preferences to uncertainty then Axioms 2 and 3 are violated.

Alternatively we can either use expected money income for both parties, or we can use expected money income for me and my expected share. In either case the independence axiom will be violated. Indeed, under the assumptions of Bolton and Ockenfels, we may find $y_1 < 1 < y_2$ so that $v(1,1/(1+y_1)) = v(1,1/(1+y_2))$. Moreover, if $s \notin \{1/(1+y_1),1/(1+y_2)\}$ then $v(1,s) \neq v(1,1/(1+y_1))$. Independence then implies the coin flip between $(1,y_1)$ and $(1,y_2)$ must be indifferent to both of the outcomes of the coin flip. The utility of the coin flip is $v(1,s)$, and since neither the share of expected income nor the expected share takes on the values $\{1/(1+y_1),1/(1+y_2)\}$, the coin flip is not indifferent two the certain outcomes.

Both the Fehr and Schmidt and the Bolton and Ockenfels preferences exhibit spite or egalitarianism in the sense that a Pareto inferior allocation may be preferred if it is fairer. In particular, for both preferences, my utility decreases in your income when $y > m$, while when $y < m$ my utility increases in your income. The former case implies a willingness for me to pay to reduce your income. This is one possible notion of fairness, but not one required in our axioms.

The remaining preferences we discuss are monotone and not egalitarian.

Charness and Rabin

The Charness and Rabin [2002] preferences are given in our notation by $U(m,y) = (1-\gamma)m + \gamma(\delta \min(m,y)) + (1-\delta)(m+y))$, where $0 \le \gamma,\delta \le 1$. That is to say, they are a weighted average of my money income, the least income either of us have, and the social total. It is the dependence on the least income of either of us that gives rise to a concern for fairness. Naturally if we extend these preferences to uncertainty using the independence axiom Axioms 2 and 3 must fail. On the other hand the extension using expected money payoffs violate independence since $\min(E\tilde{m}, E\tilde{y})$ obviously does so.

Cox and Sadiraj

Cox and Sadiraj [2004] specify preferences that depend on who is getting more in a non-linear way. In our notation, their preferences are given by

$$U(m,y) = (1 - \theta_{-}1(m < y) - \theta_{+}1(m \geq y))m^{\alpha} + (\theta_{-}1(m < y) + \theta_{+}1(m \geq y))y^{\alpha}$$

where $0 < \alpha < 1, 0 \leq \theta_{+} < 1, 0 \leq \theta_{-} \leq \theta_{+}, \theta_{-} \leq 1 - \theta_{+}$. The interpretation is that the weights on m^{α}, y^{α} depend on how fair the allocation is. Because of decreasing marginal utility in m, y, it is natural to interpret this as an expected utility function, in which case Axioms 2 and 3 are violated .

Andreoni and Miller

Andreoni and Miller [2002] consider preferences over m, y of the form $U(m,y) = \text{sgn}(\alpha)[m^{\alpha} + \delta y^{\alpha}]$, $\alpha \leq 1$. As in the case of Cox and Sadiraj, it is difficult to interpret this other than as a Von-Neumann-Morgenstern utility function so Axiom 2 and 3 are violated. Notice in the limiting Leontief case when $\delta = 1$ this model is a special case of Charness and Rabin.

Notice also that since $U(m,y)$ is additively separable it can be seen as reflecting altruism as opposed to egalitarianism *per se*. The Charness-Rabin preferences are a convex combination of two limiting cases of the Andreoni and Miller preferences ($\alpha = 1$ and $\alpha \to -\infty$) so are a limit of additively separable preferences. The Cox and Sadiraj preferences are not separable, but also are strictly increasing in m and y and so are not egalitarian.

4. **Ex Post** *Fairness and Risk Preference*

The models discussed in the previous section specify social preferences only over certain outcomes and are representable by means of a utility function. If we extend preferences to lotteries by taking the expected utility the independence axiom is necessarily satisfied, and so very weak notions of *ex ante* fairness for both you and me must fail. As we noted, an alternative procedure for extending preferences to lotteries is to replace certain income with its expected value. This allows for *ex ante* fairness, while the expected-utility extension does not. However, this method of relaxing independence does not allow for *ex post* fairness, and more broadly any preferences that have well defined risk preferences for me do not allow *ex post* fairness.

In particular, consider

Axiom 4 [*Ex Post* Fairness]: *There exist* $m_1 < m_2$ *and* $y_1 < y_2$ *such that* $((m_1, y_1),(m_2, y_2)) \succ ((m_2, y_1),(m_1, y_2))$.

This says that for some lottery the more egalitarian coin flip where we are both better off at the same time is preferred to the less egalitarian coin flip where our fates are opposite. For example, given a choice between $(1,1,0,0)$ and $(1,0,0,1)$ many people might prefer $(1,1,0,0)$. This captures the common observation that "misery likes company" as well as ideas of status competition and relative consumption.[8]

Taking, say, the Fehr-Schmidt preferences, and extending them to lotteries using expected income, allows for ex-post fairness but has the perhaps undesirable implication that agents are risk neutral; we next examine the extent to which a concern for *ex post* fairness is consistent with (a) well defined notions of risk aversion and (b) a concern for *ex ante* fairness.

Axiom 5 [Independent Individual Risk Preference]: *If* $((m_1, y_1),(m_2, y_2)) \succ ((m_3, y_1),(m_4, y_2))$ *then for all* (y_3, y_4) *we have* $((m_1, y_3),(m_2, y_4)) \succ ((m_3, y_3),(m_4, y_4))$.

This says that if we hold fixed the outcomes for you, then my risk preferences are independent of the particular outcomes you are assigned, in effect I have well-defined preferences over lotteries for me that are independent of what the lotteries give to you. One example of preferences that satisfy Axiom 5 is the utility representation obtained by Grant et al [2010] for an impartial observer:

$$U(\tilde{m}, \tilde{y}) = v_m(Eu_m(\tilde{m})) + v_y(Eu_y(\tilde{y})),$$

[8] Harel, Safra and Segal [2005], Fleubaey [2010], and Grant et al [2010] consider notions of ex-post fairness for an outside observer in settings where preferences depend only on the expected individual utilities; their conditions are assumed to hold at "many" lotteries as opposed to our condition which must hold at least one.

where v_m and v_y are both continuous and increasing This is consistent with me having well-defined risk preferences over own consumption, holding fixed the lottery of you; it also implies that my preferences over lotteries for you respects your own risk preferences. The Grant et al utility function allows for a form of *ex ante* fairness, as when the v's are concave, I am more willing to give you utility when I am better off. However, it is not consistent with a descriptive model of social preferences, as it rules out any concern *for ex post* fairness.

More generally, any preferences with a utility representation of the form $U(\tilde{m},\tilde{y}) = V(Eu_m(\tilde{m}), Eu_y(\tilde{y}))$, with V increasing in its first argument, satisfies Axiom 5, additive separability is not necessary. Moreover, preferences can satisfy both Axiom 5 and also satisfy *ex ante* fairness for both you and me, and in particular the Grant et al preferences can have this property.[9] In addition, if the Fehr and Schmidt utility function is extended to uncertainty by taking the expected value of income, then it satisfies Axiom 5. What is perhaps less transparent is that under a very mild state-independence condition, any preferences that satisfy Axiom 5 violate ex-post fairness.

Axiom 6 [State Independence]: $((m_1,y_1),(m_2,y_2)) \sim ((m_2,y_2),(m_1,y_1))$.

Theorem 3: *There are no transitive preferences satisfying Axioms 4, 5 and 6.*

Proof: By Axiom 4, let $((m_1,y_1),(m_2,y_2)) \succ ((m_2,y_1),(m_1,y_2))$. Axiom 5 implies $((m_1,y_2),(m_2,y_1)) \succ ((m_2,y_2),(m_1,y_1))$. But $((m_2,y_2),(m_1,y_1)) \sim ((m_1,y_1),(m_2,y_2))$ and $((m_1,y_2),(m_2,y_1)) \sim ((m_2,y_1),(m_1,y_2))$ by Axiom 6. We conclude that $((m_2,y_1),(m_1,y_2)) \succ ((m_1,y_1),(m_2,y_2))$, which contradicts transitivity.

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[9] This is true for example if individual utility functions are $u_m(m) = m, u_y(y) = y$, with v_m strictly increasing, $v_m'(M) \to 0$ as $M \to \infty$ and v_y strictly increasing. To see why Axiom 2 is satisfied, the assumptions guarantee the existence of an $(X,Y) \sim (1,0)$ with $X < 1$ and $Y > 0$. Consider the coin flips $A = ((1,0),(M,-1)$ and B=$((X,Y),(M,-1)$. Compute $U(A) = v_m(.5+.5M) + v_y(-.5)$, $U(B) = v_m(.5(M+X)) + v_y(.5Y-.5)$. Since $v_m'(M) \to 0$ as $M \to \infty$ also $v_m(.5+.5M) - v_m(.5(M+X)) \to 0$. On the other hand while $v_y(.5Y-.5) - v_y(-.5) > 0$. Thus when M is large enough B is strictly preferred. For Axiom 3, consider $(X,Y) \sim (1,0)$ with $X > 1$ and $Y < 0$. Then as $M \to \infty$ we again have $v_m(.5+.5M) - v_m(.5(M+X)) \to 0$ and now $v_y(.5Y-.5) - v_y(-.5) < 0$ so A is strictly preferred.

Roughly speaking the situation is this. Given preferences for fairness under certainty $U(m,y)$, their extension to lotteries by taking expected utility $Eu(\tilde{m},\tilde{y})$ must violate *ex ante* fairness for both me and you. If instead we extend the preferences to lotteries by taking the expected value of individual income $u(E\tilde{m},E\tilde{y})$ or taking the expected value of any function of individual income, we violate *ex post* fairness. We can view the former extension as exhibiting a preference for *ex post* fairness only and the second as exhibiting a preference for *ex ante* fairness only. By combining the two, we may easily get preferences for both *ex post* and *ex ante* fairness, though if Axiom 6 is satisfied Axiom 5 must be violated.

Consider, for example, defining $U(m,y)$ to be the Fehr and Schmidt functional form, and extend these to uncertainty using $U(\tilde{m},\tilde{y}) \equiv \gamma EU(\tilde{m},\tilde{y}) + (1-\gamma)U(E\tilde{m},E\tilde{y})$. (Notice that this reduces to the original utility function for deterministic outcomes.) Recall that Axioms 2 and 3 involve strict preference and we already showed $U(E\tilde{m},E\tilde{y})$ satisfies Axioms 2 and 3 for $\alpha > 0$. It follows that for γ sufficiently small, $U(\tilde{m},\tilde{y})$ must also satisfy Axioms 2 and 3. Axiom 6 is obviously satisfied, so we shall show that Axiom 4 is satisfied even when γ is small. Consider in particular $m_1 < m_2$ and $y_1 < y_2$ where $m_1 = y_1, m_2 = y_2$. Then

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\begin{aligned}
&U((m_1,y_1),(m_2,y_2)) - U((m_2,y_1),(m_1,y_2)) \\
&= (\gamma/2)(U(m_1,m_1) + U(m_2,m_2) - U(m_2,m_1) - U(m_1,m_2)) \\
&= (\gamma/2)(\alpha+\beta)(m_2 - m_1) > 0
\end{aligned}
$$

so that Axiom 5 is indeed satisfied provided that $\gamma > 0$ and $\alpha > 0$.

5. Conclusion

Our first point is that the independence axiom is inconsistent with even very weak notions of *ex ante* fairness. Existing models of fairness do not focus on the role of lotteries, and the preferences analyzed in Fehr and Schmidt [1999], Bolton and Ockenfels [2009], Charness and Rabin [2002], Cox and Sadiraj [2004] and Andreoni and Miller [2002] are defined for certain outcomes, without specifying how they are to be extended to lotteries. As we have seen, there are two obvious ways of making the extension. We can do so by treating the certainty utility as a Von Neumann-Morgenstern utility function, in which case the independence axiom is satisfied, but then *ex ante* fairness for both me and you are violated. Or the preferences can be extended by replacing the certain individual utility with its expected value, in which case the independence axiom is violated. In the later case, for example, in the social welfare theory of Grant et al [2010], not only is independence violated, but *ex post* fairness is ignored. Hence our second conclusion: the standard models of fairness and social preference used in experimental research and their obvious extensions to lotteries do not incorporate both *ex ante* and *ex post* fairness. As we have seen, it is relatively straightforward to construct preferences that do satisfy both conditions, in part because we deliberately formulated very weak notions of fairness to make the impossibility results more sharp.

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