EMBEDDING AN ANALYTIC EQUIVALENCE RELATION IN THE TRANSITIVE CLOSURE OF A BOREL RELATION

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Abstract. The transitive closure of a reflexive, symmetric, analytic relation is an analytic equivalence relation. Does some smaller class contain the transitive closure of every reflexive, symmetric, closed relation? An essentially negative answer is provided here. Every analytic equivalence relation on an arbitrary Polish space is Borel embeddable in the transitive closure of the union of two smooth Borel equivalence relations on that space. In the case of the Baire space, the two smooth relations are closed and the embedding is homeomorphic.

1. Introduction

This note answers a question in descriptive set theory that arises in the context of the Bayesian theory of decisions and games. It concerns the notion of common knowledge, formalized by Robert Aumann [1976]. For an event A that is represented as a subset of a measurable space Ω, Aumann defines the event that an agent *knows* A to be the event $A \setminus [\Omega \setminus A]_{\mathcal{P}}$, where $\mathcal{P}$ is the agent's *information partition* of Ω.[1] If $\mathcal{P}$ is the meet of individual agents' information partitions (in the lattice of partitions where $\mathcal{P}' \leq \mathcal{P}'' \iff \mathcal{P}''$ refines $\mathcal{P}'$), then Aumann defines

$$A \setminus [\Omega \setminus A]_{\mathcal{P}} \tag{1}$$

to be the event that A is common knowledge among the agents.[2]

Aumann restricts attention to the case that Ω is countable (or that the Borel σ-algebra on Ω is generated by the elements of a countable partition), so that measurability issues do not arise. But, otherwise, measurability problems dictate that information partitions should be represented as equivalence relations. If E_1 and E_2 are $\boldsymbol{\Sigma}^1_1$ (that is, analytic) equivalence relations, then the meet of the partitions

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[1] $[A]_{\mathcal{P}}$ denotes $\bigcup\{\pi \mid \pi \in \mathcal{P}$ and $\pi \cap A \neq \emptyset\}$, the saturation of A with respect to $\mathcal{P}$. If E is an equivalence relation, then $[A]_E$ denotes the saturation of A with respect to the partition induced by E. Aumann's definition corresponds to the truth condition for $\Box A$ in Kripke [1959].

[2] Aumann sketches an argument—reminiscent of a general principle in proof theory [Pohlers, 2009, Lemma 6.4.8, p. 89]—that this definition is equivalent to the intuitive, recursive definition of common knowledge: that A has occurred and that, for all $n \in \mathbb{N}$, both agents know...that both agents know (n times) that A has occurred.

that they induce is induced by the transitive closure of their union. This transitive closure is also a $\mathbf{\Sigma}^1_1$ equivalence relation.[3]

In most applications to Bayesian decision theory and game theory, it is reasonable to specify each agent's information as a $\mathbf{\Delta}^1_1$ (that is, Borel) equivalence relation, or even as a smooth Borel relation or a closed relation rather than as an arbitrary $\mathbf{\Sigma}^1_1$ equivalence relation.[4] Thus it may be asked: if the graphs of E_1 and E_2 are in $\mathbf{\Delta}^1_1$ or in some smaller class, then how is the graph of the transitive closure of $E_1 \cup E_2$ restricted?

It will be shown here that no significant restriction of the common-knowledge partition is implied by such restriction of agents' information partitions. This finding is not surprising, since restricting the complexity of individuals' equivalence relations does not obviate the use of an existential quantifier to define the transitive closure of a relation. Nevertheless, it needs to be shown that common-knowledge equivalence relations derived from Borel equivalence relations are not lower in set-theoretic complexity, as a class, than their definition would suggest.[5] Moreover, proposition 3 will show that being the union of finitely many (in fact, of fewer that $2^{\aleph_0}$) Borel equivalence relations—that is, representability in the form, of which the transitive closure is an equivalence relation specifying common-knowledge—is a stronger property than representation as the transitive closure of an arbitrary Borel, reflexive, symmetric relation.

To define the transitive closure of $R \subseteq \Omega \times \Omega$, let $R^{(1)} = R$ and $R^{(n+1)} = RR^{(n)}$ (that is, the composition of relations R and $R^{(n)}$). Denote the transitive closure of R by $R^+ = \bigcup_{n \in \mathbb{N}_+} R^{(n)}$. It will be proved here that, if Ω is a Polish space and $E_0 \subset \Omega \times \Omega$ is a $\mathbf{\Sigma}^1_1$ equivalence relation, then there are smooth $\mathbf{\Delta}^1_1$ equivalence relations E_1 and E_2 and a $\mathbf{\Delta}^1_1$ subset Z of Ω, such that $(E_1 \cup E_2)^+ \restriction Z$ is Borel equivalent to E_0.[6] If Ω is the Baire space, then E_1 and E_2 can be taken to be closed, Z can be taken to be open, and the Borel equivalence can be taken to be a homeomorphic equivalence.

2. The case of the Baire space

First take Ω to be the Baire space, $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$.[7] Define subsets X and Y of $\mathcal{N}$ by $X = \{\alpha | \alpha_0 > 0\}$ and $Y = \{\alpha | \alpha_0 = 0\}$. X and Y are both homeomorphic to $\mathcal{N}$, and homeomorphisms $f \colon X \to Y$ and $g \colon Y \times Y \times Y \to Y$ are routine to construct.[8] Each of X and Y is both open and closed in $\mathcal{N}$. It follows that, if Z is either X or Y, then $A \subseteq Z$ is open (resp. closed, Borel, $\mathbf{\Sigma}^1_1$) as a subset of A iff it is open

[3] Composition is defined with a single existential quantifier, and thus takes a pair of $\mathbf{\Sigma}^1_1$ relations to a $\mathbf{\Sigma}^1_1$ relation. The countable union of $\mathbf{\Sigma}^1_1$ relations is $\mathbf{\Sigma}^1_1$. [Moschovakis, 2009, Theorem 2B.2, p. 54]

[4] Smoothness (also called tameness) and closedness are co-extensive for equivalence relations on standard Borel spaces. [Harrington et al., 1990, proof of Theorem 1.1, p. 920] Standard Borel spaces are defined below, in section 4.

[5] If the graph of a function is an analytic set, then it is a Borel set. [Moschovakis, 2009, exercise 2E.4] This is an example of a class of Borel sets, the definition of which does not have a syntactic form that overtly excludes non-Borel analytic sets from the class.

[6] $R \restriction Z = R \cap (Z \times Z)$. Let restriction take precedence over Boolean operations. For example, $X \cup R \restriction Z \cap Y$ means $X \cup (R \restriction Z) \cap Y$.

[7] $\mathbb{N} = \{0, 1, \ldots\}$. $\mathcal{N}$ is topologized as the product of discrete spaces.

[8] Since Y is homeomorphic with $\mathcal{N}$, g can be constructed from the function described by Moschovakis [2009, p. 31].

(resp. closed, Borel, $\boldsymbol{\Sigma}^1_1$) as a subset of Z. This invariance to the ambient space extends to product spaces. (For example a subset of $X \times Y$ is closed in $X \times Y$ iff it is closed in $\mathcal{N} \times \mathcal{N}$.) In subsequent discussions, subsets of these subspaces will be characterized (for example, as being closed) without mentioning the subspace.

Theorem 1. *If $E \subseteq X \times X$ is a $\boldsymbol{\Sigma}^1_1$ equivalence relation, then there are equivalence relations I and J on $\mathcal{N} \times \mathcal{N}$, each of which has a closed graph, such that $E = (I \cup J)^+ \restriction X$.*

Before proceeding to the proof of this theorem, note that $I \cup J$ is a closed, reflexive, symmetric relation. Thus, theorem 1 has the following corollary.

Corollary 2. *If $E \subseteq X \times X$ is a $\boldsymbol{\Sigma}^1_1$ equivalence relation, then there is a closed, reflexive, symmetric relation R on $\mathcal{N} \times \mathcal{N}$, such that $E = R^+ \restriction X$.*

Theorem 1 would follow from corollary 2 if every closed, reflexive, symmetric relation were the union of two closed equivalence relations, but that is not the case. Denote the diagonal (that is, identity) relation in $\mathcal{N} \times \mathcal{N}$ by $D = \{(\alpha, \alpha) | \alpha \in \mathcal{N}\}$. D is closed.

Proposition 3. *Let $\alpha \in \mathcal{N}$. Define $R = D \cup (\{\alpha\} \times \mathcal{N}) \cup (\mathcal{N} \times \{\alpha\})$, and define*

$$\mathcal{E} = \bigcup \{(D \cup \{(\alpha, \beta), (\beta, \alpha)\}) | \beta \in \mathcal{N} \setminus \{\alpha\}\}.$$

$R = \bigcup \mathcal{E}$; every $E \in \mathcal{E}$ is an equivalence relation; R is closed, reflexive, and symmetric; and $2^{\aleph_0}$ is the cardinality of $\mathcal{E}$. There is no other set $\mathcal{F}$ of equivalence relations such that $R = \bigcup \mathcal{F}$. Thus, R is not a union of fewer that $2^{\aleph_0}$ equivalence relations.

Proof. The assertions regarding $\mathcal{E}$ are obvious from its construction. To obtain a contradiction from supposing that $\mathcal{E}$ were not unique, suppose that R were also the union of a set $\mathcal{F} \neq \mathcal{E}$ of equivalence relations. Not $\mathcal{F} \subsetneq \mathcal{E}$. So, there must be some $E \in \mathcal{F} \setminus \mathcal{E}$. By symmetry, there must be three distinct points, α, β, γ such that $\{(\beta, \alpha), (\alpha, \gamma)\} \subseteq E$. Since E is transitive, $(\beta, \gamma) \in E \setminus R$, contrary to $R = \bigcup \mathcal{F}$. □

3. Proof of the theorem

If $1 \leq i < j \leq k$ and $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathcal{N}^k$, then a transposition mapping is defined by $t_{ij}(\vec{\alpha}) = (\alpha_1, \ldots, \alpha_{i-1}, \alpha_j, \alpha_{i+1}, \ldots, \alpha_{j-1}, \alpha_i, \alpha_{j+1}, \ldots, \alpha_k)$.[9] The abbreviation $\widetilde{A} = t_{12}(A) = \{t_{12}(\alpha) | \alpha \in A\}$ will sometimes be used. Each t_{ij} is a homeomorphism of $\mathcal{N}^k$ with itself. Note that $t_{ij} \restriction X$ and $t_{ij} \restriction Y$ map X^k and Y^k homeomorphically onto themselves.

Recall that a relation $E \subseteq X \times X$ is $\boldsymbol{\Sigma}^1_1$ iff there is a set F such that

(2) $\qquad F \subseteq X \times X \times \mathcal{N}$ is closed, and $(\alpha, \beta) \in E \iff \exists \gamma\ (\alpha, \beta, \gamma) \in F$.

Lemma 4. *If $E \subseteq X \times X$ is symmetric, then E is $\boldsymbol{\Sigma}^1_1$ iff there is a closed, t_{12}-invariant set $F \subset X \times X \times X$ that satisfies (2).*

Proof. Let F_0 satisfy (2). Let h be a homeomorphism from $\mathcal{N}$ to X, and define $F_1 \subseteq X \times X \times X$ by $(\alpha, \beta, \gamma) \in F_0 \iff (\alpha, \beta, h(\gamma)) \in F_1$. F_1 also satisfies (2), then, and it is closed. By symmetry of E, $\widetilde{F_1}$ is another closed set that satisfies (2). Consequently, $F = F_1 \cup \widetilde{F_1}$ is a t_{12}-invariant closed set that satisfies (2). □

[9] A sub-sequence of subscripted alphas distinct from α_i and α_j having subscripts that are not increasing, which occurs if $i = 1$ or $j = i + 1$ or $j = k$, denotes the empty sequence.

Let i denote the identity function on $\mathcal{N}$. If K, L, M, N are any sets, and $p\colon K \to L$ and $q\colon M \to N$, then denote the product mapping by $p \times q\colon K \times M \to L \times N$.

The two closed equivalence relations that theorem 1 asserts to exist are defined from the homeomorphisms f and g introduced in section 2, and the closed, t_{12}-invariant set F guaranteed to exist by lemma 4, as follows.

$$\begin{aligned} j(\alpha,\beta,\gamma) &= g(f(\alpha), f(\beta), f(\gamma)). \\ G &= \{(\alpha, j(\alpha,\beta,\gamma)) | (\alpha,\beta,\gamma) \in F\} \subseteq X \times Y; \\ H &= \{(j(\alpha,\beta,\gamma), j(\beta,\alpha,\gamma)) | (\alpha,\beta,\gamma) \in X \times X \times X\}; \\ I &= D \cup G \cup \widetilde{G} \cup \widetilde{G}G; \\ J &= D \cup H. \end{aligned} \tag{3}$$

Lemma 5. *D, G, $\widetilde{G}$, H, and J are closed.*

Proof. D is closed because $\mathcal{N}$ is a metric space.

The function $i \times j$ is a homeomorphism from $X \times X \times X \times X$ to $X \times Y$. Being a homeomorphism, it is an open mapping (which takes closed sets to closed sets). $G = [i \times j]((D \restriction X \times X \times X) \cap (X \times F))$. $D \restriction X \times X \times X$ and $X \times F$ are both closed subsets of $X \times X \times X \times X$, so G is closed. $\widetilde{G}$ is closed, as the image of G under t_{12}, a self-homeomorphism of $\mathcal{N} \times \mathcal{N}$.

$j \times j$ is a homeomorphism from $(X \times X \times X) \times (X \times X \times X)$ to $Y \times Y$. The image under $j \times j$ of a closed subset of its domain is therefore closed in its range. $\{((\alpha,\beta,\gamma),(\beta,\alpha,\gamma)) | (\alpha,\beta,\gamma) \in X \times X \times X\}$ is $t_{23} \circ t_{25}(D \restriction X \times D \restriction X \times D \restriction X)$, which is closed. H, the image of this set under $j \times j$, is therefore closed.

J, the union of two closed sets, is closed. □

Lemma 6. *$G\widetilde{G} = D \restriction X$. $\widetilde{G}G = \{(j(\alpha,\beta,\gamma), j(\alpha,\delta,\epsilon)) | (\alpha,\beta,\gamma) \in F$ and $(\alpha,\delta,\epsilon) \in F\}$. $H = \widetilde{H}$. $H^{(2)} = D \restriction Y$. $GH = \{(\alpha, j(\beta,\alpha,\gamma) | (\alpha,\beta,\gamma) \in F\}$. $GH\widetilde{G} = E$. $\widetilde{G}G$ and I are closed.*

Proof. All assertions except the one regarding closedness of $\widetilde{G}G$ and I are verified by straightforward calculations. That F is invariant under t_{12} is used to show that $H = \widetilde{H}$ and that $E \subseteq GH\widetilde{G}$.

The proof that $\widetilde{G}G$ is closed is parallel to the proof that H is closed. According to the first part of this lemma, $\widetilde{G}G = [j \times j](t_{24}(D \restriction X \times X \times X \times X \times X) \cap (F \times F))$.

I, the union of four closed sets, is closed. □

Lemma 7. *I and J are equivalence relations.*

Proof. These relations are reflexive and symmetric, so their transitive closures are equivalence relations. Thus, the lemma is equivalent to the assertion that $I = I^+$ and $J = J^+$. For any relation K, $K^{(2)} = K$ is sufficient for $K = K^+$. In the following calculations of $I^{(2)}$ and $J^{(2)}$, composition of relations is distributed over unions. Terms that evaluate by identities that were calculated in lemma 6 to a previous term or its sub-relation, are omitted from the expansion by terms in the pentultimate step of each calculation.

$$
\begin{aligned}
I^{(2)} &= (D \cup G \cup \widetilde{G} \cup \widetilde{G}G)(D \cup G \cup \widetilde{G} \cup \widetilde{G}G) \\
&= (D \cup G \cup \widetilde{G} \cup \widetilde{G}G) \cup (G \cup G\widetilde{G} \cup G\widetilde{G}G) \cup (\widetilde{G} \cup \widetilde{G}G \cup \widetilde{G}\widetilde{G} \cup \cup \widetilde{G}\widetilde{G}G) \\
&\quad \cup (\widetilde{G}G \cup \widetilde{G}GG \cup \widetilde{G}G\widetilde{G} \cup \widetilde{G}G\widetilde{G}G) \\
&= D \cup G \cup \widetilde{G} \cup \widetilde{G}G \\
&= I.
\end{aligned} \tag{4}
$$

$$
\begin{aligned}
J^{(2)} &= (D \cup H)(D \cup H) \\
&= (D \cup H) \cup (H \cup H^{(2)}) \\
&= D \cup H \\
&= J.
\end{aligned}
$$

□

Proof of theorem 1. Lemmas 5–7 show that the each of the relations I and J on $\mathcal{N} \times \mathcal{N}$, is an equivalence relation that has a closed graph. It remains to be shown that that $E = (I \cup J)^{+} \cap (X \times X)$. Note that, since $D \subseteq I \cup J$, $I \cup J \subseteq (I \cup J)^{(2)} \subseteq (I \cup J)^{(3)} \subseteq \dots$ Hence, if $(I \cup J)^{(n)} = (I \cup J)^{(n+1)}$, then $(I \cup J)^{(n)} = (I \cup J)^{+}$.

The following calculation shows that $(I \cup J)^{(5)} = (I \cup J)^{(6)}$. The calculation is done recursively, according to the following recipe at each stage $n > 1$:

(1) Begin with the equation $(I \cup J)^{(n+1)} = (I \cup J)(I \cup J)^{(n)}$.
(2) Rewrite $(I \cup J)$ as $D \cup G \cup \widetilde{G} \cup \widetilde{G}G \cup H$ according to (3), rewrite $(I \cup J)^{(n)}$ according to the result of the previous stage, and then distribute composition of relations over union in the resulting equation.
(3) For each identity stated in lemma 6, and for each identity that, for some $K \in \{G, \widetilde{G}, H\}$, equates a composition KD or DK of K and D (or a restriction of D to a product set of which K is a subset) to K, do as follows: Going from left to right, apply the identity wherever possible.[10] Repeat this entire step (consisting of one pass per identity) until no further simplifications are possible.
(4) Delete compositions of relations that include terms KL such that the range of K and the domain of L (viewed as correspondences) are disjoint, in which case the term denotes the empty relation. Delete $D \restriction X$ (occurring as a term by itself), of which D is a superset.
(5) Delete each term of form $[K]\widetilde{G}[L]$ (resp. $[K]G[L]$) from a union in which the corresponding term for its superset, $[K]\widetilde{G}E[L]$ (resp. $[K]EG[L]$) also appears. (One or both of the bracketed sub-terms may be absent from both terms in the pair.) Delete D (occurring as a term by itself) from every union that contains both $D \restriction Y$ and E, since $D \subseteq D \restriction Y \cup E$.

[10] Let $P = D \restriction X$ and $Q = D \restriction Y$. Identities are applied in the following order at each stage of the recursion: $DD = D$, $DE = E$, $DG = G$, $D\widetilde{G} = \widetilde{G}$, $DH = H$, $DP = P$, $DQ = Q$, $ED = E$, $EE = E$, $EP = E$, $GD = G$, $G\widetilde{G} = P$, $GH\widetilde{G} = E$, $GQ = G$, $\widetilde{G}D = \widetilde{G}$, $\widetilde{G}P = \widetilde{G}$, $HD = H$, $HH = Q$, $HQ = H$, $PD = P$, $PE = E$, $PG = G$, $PP = P$, $QD = Q$, $Q\widetilde{G} = \widetilde{G}$, $QH = H$, $QQ = Q$.

(6) Reorder terms lexicographically, in the order $D < D \restriction Y < E < G < \widetilde{G} < H$. Delete repeated terms.

$$
\begin{aligned}
(I \cup J) &= D \cup G \cup \widetilde{G} \cup \widetilde{G}G \cup H \\
(I \cup J)^{(2)} &= D \cup D \restriction Y \cup G \cup GH \cup \widetilde{G} \cup \widetilde{G}G \cup \widetilde{G}GH \\
&\quad \cup H \cup H\widetilde{G} \cup H\widetilde{G}G \\
(I \cup J)^{(3)} &= D \restriction Y \cup E \cup EG \cup GH \cup \widetilde{G}E \cup \widetilde{G}EG \cup \widetilde{G}GH \\
&\quad \cup H \cup H\widetilde{G} \cup H\widetilde{G}G \cup H\widetilde{G}GH \\
(I \cup J)^{(4)} &= D \restriction Y \cup E \cup EG \cup EGH \cup \widetilde{G}E \cup \widetilde{G}EG \cup \widetilde{G}EGH \\
&\quad \cup H \cup H\widetilde{G}E \cup H\widetilde{G}EG \cup H\widetilde{G}GH \\
(I \cup J)^{(5)} &= D \restriction Y \cup E \cup EG \cup EGH \cup \widetilde{G}E \cup \widetilde{G}EG \cup \widetilde{G}EGH \\
&\quad \cup H \cup H\widetilde{G}E \cup H\widetilde{G}EG \cup H\widetilde{G}EGH \\
&= (I \cup J)^{(6)}
\end{aligned} \tag{5}
$$

Thus $(I \cup J)^+ = (I \cup J)^{(5)}$. Note that $D \restriction Y$, G, $\widetilde{G}$, H and all relations of form or $\widetilde{G}Q$ or HQ or QG or QH (where variable Q ranges over compositions of G, $\widetilde{G}$, H, and E), are disjoint from $X \times X$. Therefore, from the calculation in (5) of $(I \cup J)^{(5)}$ as a union of E with such relations, it follows that $(I \cup J)^+ \restriction X = E$. □

4. The general case of a standard Borel space

In this concluding section, theorem 1 is generalized in two ways to an arbitrary standard Borel space. A *standard Borel space* is a pair $\mathbf{\Omega}_0 = (\Omega_0, \mathcal{B}_0)$ such that, for some pair $\mathbf{\Omega} = (\Omega, \mathcal{B})$, $\mathcal{B}$ is the σ-algebra of Borel subsets of the set Ω under some Polish topology, $\Omega_0 \in \mathcal{B}$, and $\mathcal{B}_0 = \{B_0 | \exists B\, [B \in \mathcal{B} \text{ and } B_0 = B \cap \Omega_0]\}$. A *Borel isomorphism* of standard Borel spaces $\mathbf{\Omega}_0$ and $\mathbf{\Omega}$ is a $\mathbf{\Delta}^1_1$ function $k \colon \Omega_0 \to \Omega$ such that $k^{-1} \colon \Omega \to \Omega_0$ exists and is also $\mathbf{\Delta}^1_1$. A $\mathbf{\Delta}^1_1$ subset of a standard Borel space is also a standard Borel space, and every two uncountable standard Borel spaces are isomorphic.[11]

In both generalizations, the concept of smoothness of a Borel equivalence relation substitutes for the concept of closedness that appears in theorem 1. If $E \subseteq \Omega \times \Omega$ is a $\mathbf{\Delta}^1_1$ equivalence relation, and if there is a set $\{Y_n\}_{n \in \mathbb{N}}$ of $\mathbf{\Delta}^1_1$ sets such that $(\omega, \omega') \in E \iff \forall n\, [\omega \in Y_n \iff \omega' \in Y_n]$, then E is a *smooth* equivalence relation. By Harrington et al. [1990, proof of Theorem 1.1, p. 920], every equivalence relation with closed graph is smooth. If $k \colon \Omega_0 \to \Omega$ is $\mathbf{\Delta}^1_1$ and $E \subseteq \Omega \times \Omega$ is a smooth $\mathbf{\Delta}^1_1$ equivalence relation, then $E_0 \subseteq \Omega_0 \times \Omega_0$ defined by $(\psi, \omega) \in E_0 \iff (k(\psi), k(\omega)) \in E$ is also smooth, with E_0-equivalence determined by $\{k^{-1}(Y_n)\}_{n \in \mathbb{N}}$.

[11] Mackey [1957, pp. 338–9]. Henceforth, $\mathcal{B}$ will be implicit and the structure $\mathbf{\Omega}$ will be identified with the set Ω on which it is defined.

The first generalization of theorem 1 asserts Borel embeddability of an arbitrary $\mathbf{\Sigma}^1_1$ equivalence relation. If Ω_0 and Ω are standard Borel spaces, and $E_0 \subseteq \Omega_0 \times \Omega_0$ and $E \subseteq \Omega \times \Omega$ are $\mathbf{\Sigma}^1_1$ equivalence relations, then a *Borel embedding* of E_0 into E is a Borel isomorphism $e \colon \Omega_0 \to Z \subseteq \Omega$ that extends naturally to a Borel isomorphism from E_0 to $E \restriction Z$. That is, $(\psi, \omega) \in E_0 \iff (e(\psi), e(\omega)) \in E$.

Corollary 8. *Let Ω_0 and Ω be standard Borel spaces, and let $E_0 \subseteq \Omega_0 \times \Omega_0$ be a $\mathbf{\Sigma}^1_1$ equivalence relation. There are smooth $\mathbf{\Delta}^1_1$ equivalence relations $E_1 \subseteq \Omega \times \Omega$ and $E_2 \subseteq \Omega \times \Omega$ such that E_0 is Borel embeddable in $(E_1 \cup E_2)^+$.*

Proof. If Ω_0 is countable, then E_1 and E_2 can both be taken to be the image of E_0 under an arbitrary injection of Ω_0 into Ω. Otherwise, there is a Borel isomorphism $k_0 \colon \Omega_0 \to X$ (where X is as in theorem 1), and there is a Borel isomorphism $k \colon \Omega \to X$. Define $e = k^{-1} \circ k_0$ and define $Z \subseteq \Omega$ by $Z = e(\Omega_0)$. If $E \subset X \times X$ is defined by $(\alpha, \beta) \in E \iff (k_0^{-1}(\alpha), k_0^{-1}(\beta)) \in E_0$, then E is a $\mathbf{\Sigma}^1_1$ equivalence relation.[12] Let I and J be the closed equivalence relations defined in (3), and define $(\psi, \omega) \in E_1 \iff (k(\psi), k(\omega)) \in I$ and $(\psi, \omega) \in E_2 \iff (k(\psi), k(\omega)) \in J$. E_1 and E_2 are smooth. Now the corollary follows immediately from theorem 1. □

The second generalization of theorem 1 applies to a $\mathbf{\Sigma}^1_1$ equivalence relation, the restriction of which to some uncountable Borel subset of Ω is the diagonal relation. That is, the set of points, the singletons of which are blocks of the partition induced by the relation, must have an uncountable $\mathbf{\Delta}^1_1$ subset.[13]

Corollary 9. *Suppose Ω is a standard Borel space and that $E \subseteq \Omega \times \Omega$ is a $\mathbf{\Sigma}^1_1$ equivalence relation such that, for some uncountable $\mathbf{\Delta}^1_1$ set $B \subseteq \Omega$, $E \restriction B = D \restriction B$. Define $\Omega_0 = \Omega \setminus B$. Then there are smooth $\mathbf{\Delta}^1_1$ relations E_1 and E_2, such that $E \setminus D \subseteq E \restriction \Omega_0 \cup D \restriction B$*

Finally, corollary 9 provides a negative answer to the question, implicit in the preceding discussion of Aumann's formulation of common knowledge of an event, of whether the saturations of Borel sets (or even of singletons) with respect to the transitive closures of unions of smooth Borel equivalence relations lie within any significantly restricted sub-class of $\mathbf{\Sigma}^1_1$.

Corollary 10. *Suppose Ω is a standard Borel space and that $S \subseteq \Omega$ is a $\mathbf{\Sigma}^1_1$ set such that, for some $\mathbf{\Delta}^1_1$ set Ω_0, $S \subseteq \Omega_0$ and $\Omega \setminus \Omega_0$ is uncountable. Then there are smooth $\mathbf{\Delta}^1_1$ relations E_1 and E_2, such that for every non-empty $A \subseteq S$, $[A]_{(E_1 \cup E_2)^+} \cap \Omega_0 = S$.*

Proof. Define $(\psi, \omega) \in E \iff [\{\psi, \omega\} \subseteq S \text{ or } \psi = \omega]$, specify $B = \Omega \setminus \Omega_0$, and apply corollary 9. For some block, π, of the partition induced by $(E_1 \cup E_2)^+$, $\pi \cap \Omega_0 = S$. Therefore, if $\emptyset \neq A \subseteq S$, then $[A]_{(E_1 \cup E_2)^+} \cap \Omega_0 = S$. □

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[12] Moschovakis [2009, Theorem 2B.2, p. 54].

[13] If E is a $\mathbf{\Sigma}^1_1$ equivalence relation, then the set of all such points is a $\mathbf{\Pi}^1_1$ subset of Ω. One sufficient condition for an uncountable $\mathbf{\Pi}^1_1$ set, W, to have an uncountable $\mathbf{\Delta}^1_1$ subset is that there should be a nonatomic measure, μ, on Ω such that $\mu^*(\Omega \setminus W) < \mu(\Omega)$ (where μ^* is outer measure). Another sufficient condition is that W should have a perfect (hence both uncountable and $\mathbf{\Delta}^1_1$) subset. A sufficient condition for every uncountable $\mathbf{\Pi}^1_1$ set to have a non-empty perfect subset—albeit one that is independent of ZFC set theory—is that $\mathbf{\Pi}^1_1$ is determinate. [Moschovakis, 2009, Exercise 6G.10, p. 288]. It is provable in ZFL that there is an uncountable $\mathbf{\Pi}^1_1$ set (in fact, a Π^1_1 set) without a non-empty perfect subset. [Moschovakis, 2009, Exercise 5A.8, p. 212].

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