

# Optimal Capital Taxation with Idiosyncratic Investment Risk

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## Abstract

We examine the optimal taxation of capital in a general-equilibrium heterogeneous-agent economy with uninsurable idiosyncratic investment or capital-income risk. Our framework combines elements from the Ramsey and the Mirrlees traditions, as we consider a linear tax but also introduce lump-sum transfers. The tractability of our model allows for an analytic characterization of the long run, as well as along the transition to the steady state. Specifically, the ex ante optimal tax, evaluated in the long run, maximizes human wealth, namely the present discounted value of agents' income from riskless sources. The sign of the optimal tax depends on the strength of distortion versus redistribution considerations. When idiosyncratic risk in the economy is below a minimum lower bound, the optimal tax is negative, brings capital to its first-best level, and does not maximize aggregate consumption. When risk is above this lower bound, the optimal tax can be either positive or negative, brings the capital stock to its maximum feasible level, and maximizes aggregate consumption. In both cases, the tax encourages risk taking and increases capital accumulation. We also show that the transition to the long-run optimal tax occurs monotonically from above, so that the short run entails higher capital taxes than the long run.

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# 1 Introduction

We study the optimal taxation of capital in a general-equilibrium heterogeneous-agent economy with uninsurable idiosyncratic investment or capital-income risk. Such risk is an empirically relevant source of market incompleteness influencing the investment decisions of a wide range of agents, such as entrepreneurs, private business owners, and managers of publicly traded firms. In this framework, capital taxation poses an interesting tradeoff: On the one hand, it comes at a cost, as it distorts agents' saving decisions. On the other hand, it has benefits, as it reduces the variance of idiosyncratic investment risk.

Our modeling framework builds on Angeletos (2007), who develops a variant of the neoclassical growth model that allows for idiosyncratic investment risk, and studies the effects of such risk on macroeconomic aggregates. In particular, agents own privately-held businesses operating under constant returns to scale and subject to idiosyncratic risk that cannot be diversified away. However, agents are not exposed to labor-income risk, and they can also freely borrow and lend in a riskless bond. Abstracting from borrowing constraints, labor-income risk and other market frictions, isolates the impact of the idiosyncratic investment risk, and preserves tractability of the model. There is a government, imposing a proportional tax on capital income, along with a non-contingent lump-sum tax or transfer. In a similar framework, Panousi (2012) performs comparative statics with respect to the capital-income tax and finds that an increase in capital taxation may actually stimulate capital accumulation.

We consider a linear capital-income tax, along the lines of the Ramsey optimal taxation tradition, but we also introduce non-contingent lump-sum taxes or transfers that allow the government to balance its budget every period, along the lines of the Mirrlees tradition. We abstract from the possibility of government debt, because Ricardian equivalence holds in our framework. This means that in our model the optimal capital tax reflects solely the government's desire to influence individual decisions or allocations, and not to collect government revenue. In other words, the lump-sum tax controls the starting point of the tax schedule, and the proportional capital tax yields the slope, similar to what happens in a Mirrleesian framework, where taxes are allowed to vary with the level of capital income (though here the marginal tax is constrained to be constant).<sup>1</sup> However, because of its Ramsey aspect, our framework has the advantage of not imposing stringent informational requirements on the social planner.

Our paper is the first in the general-equilibrium Ramsey literature of heterogeneous agents and incomplete markets to provide an analytic characterization of the optimal tax in

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<sup>1</sup>For our model, using a Mirrlees framework would not be particularly interesting, as it would simply allow us to restore first best by taxing the excess private returns on investment.

the long run and along the transition to the steady state. First, the ex ante optimal tax, evaluated in the long run, always maximizes human wealth, namely the present discounted value of agents' income from sources that are not subject to risk. The sign of the optimal tax depends on the strength of distortion versus insurance or redistribution considerations. When idiosyncratic risk in the economy is below a minimum lower bound, the economy is sufficiently "close" to its complete-markets counterpart and redistribution considerations are relatively weak. In this case, the optimal tax is negative, brings capital to its first-best level, and does not maximize aggregate consumption. When risk is above this lower bound, markets are sufficiently incomplete and redistribution considerations are relatively strong. In this case, the optimal tax can be either positive or negative, brings the capital stock to its maximum feasible level given the economy's constraints, and maximizes aggregate consumption. Furthermore, when risk is above a certain upper bound, then the optimal tax is necessarily positive. In all cases, the optimal tax encourages risk taking and increases capital accumulation. Second, we show analytically that the transition to the long-run optimal tax occurs monotonically from above, so that the short run entails higher capital taxes than the long run. This is essentially due to general-equilibrium effects of the tax on asset returns, which outweigh the partial-equilibrium distortionary effect of the tax on the mean return to saving.

In order to elucidate the intuition behind our main results, it is useful to compare our mechanisms to those in Aiyagari's (1995) seminal optimal taxation paper with labor-income risk. The mechanism in our model is different from Aiyagari (1995) in two meaningful ways. First, as shown in Angeletos (2007), uninsurable capital-income risk may lead to underaccumulation of capital, compared to the first best, if the standard incomplete-markets precautionary saving effect is dominated by the negative effect on investment induced by the risk premium agents require to invest in capital. This is in contrast to Aiyagari (1995), where steady-state capital is always higher than at the first best. In fact, Angeletos (2007) shows that, for empirically plausible parameter values, steady-state capital is below its first-best level. In particular, for the case of logarithmic preferences, characterized by an elasticity of intertemporal substitution of one, which is the relevant empirical value for investors and a critical parameter in the model, there is always underaccumulation of capital. For purposes of expositional tractability and empirical relevance, our baseline theoretical model therefore focuses on the case of logarithmic utility. Second, as shown in Panousi (2012), capital may be increasing with the tax, due to general-equilibrium insurance effects resulting from the adjustment of the safe rate. By contrast, in Aiyagari (1995) capital unambiguously falls with the tax. These insurance effects in our model are stronger when the variance of idiosyncratic

risk is higher.

From these mechanisms it then follows that in Aiyagari (1995) the optimal tax is positive, so as to bring capital down to its first-best level. In our model, the sign of the optimal tax is ambiguous, but the rationale for optimal taxation has a similar flavor, in that the optimal tax aims to bring steady-state capital as close to the first best as possible. However, getting all the way to the first best is not always feasible. If capital risk is low enough, then the tax brings capital all the way to the first best: Capital is decreasing with the tax and the optimal tax is a subsidy, which induces capital accumulation. If risk is high, then the tax does not bring the capital stock all the way to the first best, but it nonetheless brings it to its maximum, given the constraints imposed by market incompleteness: The relationship between the tax and capital is non-monotone, and the optimal tax can be either positive or negative, depending on the region where the maximum for capital is achieved.

Hence, the optimal tax always aims to increase investment, whereas in Aiyagari (1995) it always aims to reduce investment. In addition, the sign of the optimal tax in our model is ambiguous, whereas in Aiyagari it is unambiguously positive. The specific mechanism through which the optimal tax performs its role of bringing capital as close as possible to the first best depends on the level of uninsurable idiosyncratic risk in the economy. When risk is very low, risk-taking is encouraged through the optimal capital subsidy, which increases the mean return to saving in partial equilibrium. When risk is sufficiently high, risk-taking is encouraged through a positive optimal tax, which, though it reduces the mean return to saving, it also reduces the effective variance of risk.

In sum, we find that the rationale for positive capital-income taxation in the long run does not necessarily extend to the case where markets are incomplete due to the presence of idiosyncratic capital-income risk. In particular, the sign of the optimal tax can be determined by reference to the degree of market incompleteness. Intuitively, for low levels of idiosyncratic risk, the aggregates respond to the tax as if markets were complete, capital is decreasing with the tax, and the optimal tax is a subsidy, since a subsidy induces capital accumulation. For high levels of risk, capital is increasing with the tax, and the optimal tax is positive, since a positive tax increases capital toward the first best.

## 2 Related Literature

We focus on an environment with idiosyncratic investment risk, because such risk is in fact empirically relevant for all investment decision makers, even in a financially developed country like the United States. Moskowitz and Vissing-Jorgensen (2002), among others,

using cross sectional data from the Survey of Consumer Finances, find that about 80 percent of all *private equity* in the US is owned by agents who are actively involved in the management of their own firm, and for whom such investment constitutes at least half of their total net worth. DeBacker, Panousi and Ramnath (2013), using data from US income-tax returns and a panel of income from privately held businesses, document that entrepreneurs face significant business-income risks over time. Panousi and Papanikolaou (2012) show that the negative relationship between idiosyncratic risk and the investment of *publicly-traded* firms in the US is stronger in firms where the managers hold a larger fraction of the firm's shares. Combined, these findings strengthen the empirical applicability of our model setup, because they demonstrate that a large fraction of total investment in the US is sensitive to idiosyncratic risk, essentially through the risk aversion of the agents making the investment decisions.

This paper relates to the macroeconomic literature on optimal taxation, and combines elements from both the Ramsey and the Mirrlees optimal taxation traditions. Most of this literature has focused either on complete markets or on incomplete markets with uninsurable labor-income risk. Methodologically, our approach is related to that in Bhandari, Evans, Golosov and Sargent (2013), who study optimal taxation in a heterogeneous agent economy where the government cares about redistribution and chooses transfers optimally. They find that the trade-off between redistribution and efficiency changes the well-known properties of optimal tax and debt policies. Similarly, though in a different framework, we find that the optimality of positive capital taxes in incomplete markets does not necessarily carry through when capital-income risk is the source of market incompleteness and the social planner has access to redistributive transfers.

Starting with the Ramsey literature of exogenously given market structure and exogenous policy instruments, Chamley (1986), Judd (1985), and Atkeson, Chari and Kehoe (1999) establish the result of zero optimal capital taxation when markets are complete. Correia (1996) shows that, in the neoclassical model, if there are restrictions on the taxation of production factors, then the tax rate on capital income in steady state is different from zero. Aiyagari (1995) extends the complete-markets framework to include uninsurable labor-income risk and borrowing constraints, and finds that the optimal capital tax is positive in the long run.<sup>2</sup> Chamley (2001) argues that the reasoning behind Aiyagari's positive optimal capital tax result is due to the ex ante insurance or ex-post redistribution aspect of the tax: The planner taxes agents with high income realizations and subsidizes agents with low

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<sup>2</sup>A related but different normative exercise is conducted by Davila, Hong, Krusell and Rios-Rull (2005), in the spirit of Geanakoplos-Polemarchakis.

income realizations, thereby equalizing consumption across different types of agents. Our paper incorporates elements from the Ramsey tradition and shows that the optimal tax will be positive or negative depending, in part, on the strength of insurance or redistribution considerations.

Moving to the Mirrlees literature of endogenous market incompleteness and endogenous policy instruments, Albanesi (2006) considers optimal taxation in a two-period model of entrepreneurial activity in a constrained-efficiency setting. The benefit of her approach is that the source of incomplete risk sharing is endogenously specified, and that there are no ad hoc restrictions on the tax instruments. However, her model does not allow for dynamics, or for general-equilibrium effects like those we study. In addition, our setup has the benefit of imposing fewer informational constraints on the planner. Shourideh (2011) also studies the optimal taxation of entrepreneurial income. In his model, as in our paper, the intertemporal wedge determining the tax on wealth cannot be unambiguously signed. However, the incentive constraint creates a force toward a wealth subsidy, since increasing capital tends to loosen the incentive compatibility constraint in the future. In general, however, the extensive theoretical work on taxation originating from the Mirrlees tradition focuses on labor-income risk.<sup>3</sup> This literature finds that, if insurance is limited due to the presence of asymmetric information, then it may be best to restrict free access to savings. This result has in turn been interpreted as a justification for positive capital taxation. Farhi and Werning (2010) study optimal nonlinear taxation of labor and capital in a political economy model with heterogeneous agents without commitment and find that capital taxation emerges as an efficient redistributive tool for addressing future inequality.

The overlapping-generations literature has often found support for positive optimal capital taxation. Conesa, Kitao and Krueger (2009) quantitatively characterize the optimal capital and labor income taxes in an overlapping generations model with idiosyncratic uninsurable income shocks and permanent productivity differences across households. They find that the optimal capital-income tax rate is significantly positive, mainly driven by the life-cycle structure of the model. Domeij and Heathcote (2004) perform a similar exercise. Erosa and Gervais (2002), in an overlapping-generations economy where agents' productivity varies over time, find that positive capital taxes may be optimal when labor taxes cannot be conditioned on age. Garriga (2003), and Peterman (2011) also find similar results.

A strand of the public finance literature has examined the effects of capital taxation on risk taking in a partial equilibrium framework. Some examples include Domar and Musgrave

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<sup>3</sup>Some examples include Diamond and Mirrlees (1978), Golosov, Kocherlakota, and Tsyvinski (2003), Albanesi and Sleet (2006), and Golosov, Troshkin, Tsyvinsky and Weinzierl (2010).

(1944), Stiglitz (1969), Ahsan (1974), Sandmo (1977), and Kanbur (1981). These authors argue that, by effectively reducing the variance of capital income, the capital tax allows for increased social risk taking, leading to an increase in investment in the risky asset (capital). Our results are of similar flavor, even in the case where the optimal tax turns out to be negative, but they reflect general equilibrium considerations that this literature cannot capture. As it turns out, a capital subsidy enhances risk taking when the amount of exogenous risk in the economy was too low to begin with. Our paper is also related to Varian (1980), who assumes that differences in observed income are due to exogenous differences in luck. In a two-period model of endogenous saving, he finds that the optimal capital-income tax is positive, due to the trade-off it involves between the distortion in the saving decision and the provision of social insurance through redistribution. By contrast, in our dynamic framework the optimal tax need not always be positive.

Our results are different from Panousi (2012) along two dimensions. First, Panousi (2012) performs comparative static exercises with respect to the capital tax, but does not consider the full optimal taxation problem of maximizing ex ante welfare, while taking transitional dynamics into account. Hence, our paper compares to Panousi the way Chamley (1986) compares to Judd (1985). Second, Panousi finds that capital taxation may increase capital accumulation and that the elimination of the capital tax would lead to welfare losses for most agents. This might suggest that the optimal capital tax will necessarily be positive. However, here we show that the optimal tax will in fact be negative when risk is not very high, in which case, for most agents, the redistribution effects of the tax are offset by its distortion effects on saving. In other words, when risk is not very high, then the effect of the tax on the mean return to saving outweighs its effect on the variance of saving for most agents, leading to a negative optimal tax.

The theoretical framework builds on a continuous-time variant of Angeletos (2007), but in addition it introduces a government, imposing proportional taxes on capital and labor income, along with a non-contingent lump-sum tax or transfer. Using a model similar to the one in the present paper, Angeletos and Panousi (2009) examine the effects of government consumption on steady state aggregates, for the case where government spending is financed solely through lump-sum taxes, while Angeletos and Panousi (2011) study the effects of financial integration for welfare and current-account dynamics.

The rest of the paper is organized as follows. Section 3 sets up the baseline model. Section 4 examines individual behavior, the general equilibrium and the steady state. Section 5 presents the planner's optimal taxation problem and characterizes the ex ante optimal tax in the long run. Section 6 characterizes the ex ante optimal tax in the short run. Section 7

concludes. All proofs are presented in detail in the appendix.

### 3 The basic model

Time is continuous and indexed by  $t \in [0, \infty)$ . There is a continuum of infinitely lived households distributed uniformly over  $[0, 1]$ . Each household consists of a worker and an investor. The worker is endowed with one unit of labor, supplied inelastically in a competitive labor market. The investor owns and runs a privately-held firm. Each firm employs labor in the competitive labor market, but can only use the capital stock invested by the particular household. Each firm is subject to idiosyncratic shocks, which the household can only partially diversify, as it cannot invest in other households' firms. However, each household can freely save or borrow in a riskless bond (up to a natural borrowing constraint), which is in zero net supply. In terms of timing for the firm's problem, first capital is installed, then the idiosyncratic shock is realized, and lastly the labor choice is made. All uncertainty is purely idiosyncratic, and therefore aggregates are deterministic. Finally, the government imposes a proportional tax on saving, applied on the risky as well as the riskless return, and balances the budget by giving back to agents, in the form of lump-sum transfers, the proceeds of taxation net of any government spending. Throughout the paper, for any variable  $y$ , the notation  $y_t$  is used as short-hand notation for  $y(t)$ , where  $t$  is time.

#### 3.1 Households, firms, and idiosyncratic risk

Preferences are logarithmic over consumption,  $c$ :

$$U_t = E_t \int_t^\infty e^{-\beta s} \ln(c_s), ds \quad (1)$$

where  $\beta > 0$  is the discount rate.

The financial wealth of a household  $i$ , denoted by  $a_t^i$ , is the sum of its asset holdings in private capital,  $k_t^i$ , and in the riskless bond,  $b_t^i$ , so that  $a_t^i = k_t^i + b_t^i$ . The evolution of  $a_t^i$  is given by the household budget:

$$da_t^i = (1 - \tau_t) d\pi_t^i + [(1 - \tau_t) R_t b_t^i + w_t + T_t - c_t^i] dt, \quad (2)$$

where  $d\pi_t^i$  are the profits from the firm the household operates or the household's capital income;  $R_t$  is the risk-free rate or interest rate on the riskless bond;  $w_t$  is the wage rate in the aggregate economy;  $c_t^i$  is consumption;  $\tau_t$  is the proportional savings or capital-income tax,

applied to the income from the capital and the bond alike; and  $T_t$  are non-contingent lump-sum transfers received from the government. A no-Ponzi-game condition is also imposed.

Firm profits are subject to undiversified idiosyncratic risk:

$$d\pi_t^i = [ F(k_t^i, l_t^i) - w_t l_t^i - \delta k_t^i ] dt + \sigma k_t^i dz_t^i. \quad (3)$$

Here,  $F$  is a constant-returns-to-scale neoclassical production function, assumed to be Cobb-Douglas for simplicity, namely  $F(k, l) = k^\alpha l^{1-\alpha}$  with  $\alpha \in (0, 1)$ , where  $l_t^i$  is the amount of labor the firms hires in the competitive labor market. In addition,  $\delta$  is the mean depreciation rate in the aggregate economy. Idiosyncratic risk is introduced through  $dz_t^i$ , a standard Wiener process that is i.i.d. across agents and across time.<sup>4</sup> The scalar  $\sigma$  measures the amount of undiversified idiosyncratic risk, and it is an index of market incompleteness, with higher  $\sigma$  corresponding to a lower degree of risk-sharing, and with  $\sigma = 0$  corresponding to complete markets.

### 3.2 Government

At each point in time the government taxes capital income and bond income at the rate  $\tau_t$ . The government also does government spending at the rate  $G_t$ , where  $G_t$  does not enter any production or utility functions. The proceeds of taxation, minus any government consumption, are then distributed back to the households in the form of non-contingent lump-sum transfers,  $T_t$ . The government budget constraint is therefore:

$$0 = [ \tau_t ( F_{K_t} ( \int_i k_t^i, 1 ) - \delta ) \int_i k_t^i - G_t - T_t ] dt, \quad (4)$$

where  $F_{K_t} ( \int_i k_t^i, 1 )$  is the marginal product of capital in the aggregate economy, and  $\int_i l_t^i = 1$ . For simplicity, we will henceforth set  $\delta = 0$  and  $G_t = 0$  for all  $t$ .<sup>5</sup>

## 4 Equilibrium and steady state

This section characterizes the equilibrium of the economy. First, it solves for households' optimal plans, given the sequences of prices and policies. It then aggregates across households to derive the general equilibrium dynamics and the steady state.

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<sup>4</sup>Literally taken,  $dz_t^i$  represents a stochastic depreciation shock. However, these shocks can also be interpreted as stochastic productivity shocks.

<sup>5</sup>None of our theoretical results hinge on this assumption.

## 4.1 Individual behavior

Investors choose employment after their capital stock has been installed and their idiosyncratic shock has been observed. Hence, since their production function,  $F$ , exhibits constant returns to scale, optimal firm employment and profits are linear in own capital:

$$l_t^i = l(w_t) k_t^i \quad \text{and} \quad d\pi_t^i = r(w_t) k_t^i dt + \sigma k_t^i dz_t^i, \quad (5)$$

where  $l(w_t) \equiv \arg \max_l [F(1, l) - w_t l]$  and  $r(w_t) \equiv \max_l [F(1, l) - w_t l] - \delta$ . Here,  $r_t \equiv r(w_t)$  is an investor's expectation of the return to his capital prior to the realization of his idiosyncratic shock, as well as the mean of the realized returns in the cross section of firms, since there is no aggregate uncertainty. As in Angeletos (2007), the key result here is that investors face linear, albeit risky, returns to their investment. To see how this translates to linearity of wealth in assets, let  $h_t$  denote a household's human wealth, namely the present discounted value of net-of-taxes labor endowment plus government transfers:

$$h_t = \int_t^\infty e^{-\int_t^s (1-\tau_j) R_j dj} (w_s + T_s) ds. \quad (6)$$

Next, define total effective wealth,  $x_t^i$ , as the sum of financial wealth and human wealth:

$$x_t^i \equiv a_t^i + h_t = k_t^i + b_t^i + h_t. \quad (7)$$

Total effective wealth is then the only state variable relevant for the household's optimization problem. The only constraint imposed is that consumption is non-negative, which implies non-negativity of total effective wealth, so that  $x_t^i \geq 0 \Leftrightarrow x_t^i \geq -h_t$ . In other words, there is no ad hoc borrowing constraint, and agents can freely borrow and lend up to the natural borrowing limit. The evolution of total effective wealth is then described by:

$$dx_t^i = [(1 - \tau_t) r_t k_t^i + (1 - \tau_t) R_t (b_t^i + h_t) - c_t^i] dt + \sigma (1 - \tau_t) k_t^i dz_t^i. \quad (8)$$

The first term in (8) captures the expected rate of growth of effective wealth, and it shows that wealth grows when saving exceeds consumption expenditures. The second term captures the effect of idiosyncratic risk. This linearity of wealth in assets, together with the homotheticity of preferences, ensures that the household's consumption-saving problem reduces to a tractable homothetic optimization problem, as in Samuelson's and Merton's classic portfolio analysis. Therefore, the optimal individual policy rules will be linear in total effective wealth, for given prices and government policies, as the next proposition shows.

**Proposition 1.** *Let  $\{w_t, R_t, r_t\}_{t \in [0, \infty)}$  and  $\{\tau_t, T_t\}_{t \in [0, \infty)}$  be equilibrium price and policy sequences. The household maximizes preferences as described in (1) subject to the total effective wealth evolution constraint (8). Formally, the value function,  $V^i$ , for a household with initial wealth  $x_0^i$  at  $t = 0$ , given the tax sequence  $\{\tau_t\}_{t=0}^\infty$ , is the solution to the problem:*

$$V^i \equiv V(x_0^i; \{\tau_t\}_{t=0}^\infty) = \max_{c^i, \phi^i} E_{-1} \int_0^\infty e^{-\beta t} \ln(c_t^i) dt \quad \text{s.t. (8)}. \quad (9)$$

The household's optimal consumption, investment, and bond holding choices, respectively, are then given by:

$$c_t^i = m_t x_t^i, \quad k_t^i = \phi_t x_t^i, \quad b_t^i = (1 - \phi_t) x_t^i - h_t, \quad (10)$$

where the fraction of effective wealth invested in capital,  $\phi_t$ , is given by:

$$\phi_t = \frac{(1 - \tau_t) r_t - (1 - \tau_t) R_t}{\sigma^2 (1 - \tau_t)^2}, \quad (11)$$

and the marginal propensity to consume,  $m_t$ , is constant and equal to the discount rate in preferences, i.e.  $m_t = \beta$  for all  $t$ .

The proof of Proposition 1 is in the appendix. The fact that investment is subject to undiversifiable idiosyncratic risk introduces a wage between the marginal product of capital and the risk-free rate, so that  $(1 - \tau_t)R_t < (1 - \tau_t)r_t$ . In other words, it has to be that, in equilibrium, the mean return to the risky asset (capital) exceeds the mean return to the safe asset (bond) by an amount equal to the positive (private) risk premium agents require as compensation for undertaking risky investment. The fraction of wealth invested in the risky asset,  $\phi_t$ , is then increasing in this risk premium, and decreasing in the effective variance of risk,  $\sigma(1 - \tau_t)$ . Furthermore,  $\phi_t$  is the same across all agents and it does not depend on the level of wealth. In equation (10), optimal consumption is a linear function of total effective wealth, where the marginal propensity to consume,  $m_t$ , is also independent of wealth. Moreover, because preferences are logarithmic,  $m_t$  is constant over time and equal to the discount rate,  $\beta$ , for all  $t$ .

The wealth evolution constraint, incorporating bond market clearing and individual optimization, is:

$$dx_t^i = [(1 - \tau_t)r_t\phi_t + (1 - \tau_t)R_t(1 - \phi_t) - \beta]x_t^i dt + \sigma(1 - \tau_t)\phi_t x_t^i dz_t^i. \quad (12)$$

Using (12) and Proposition 1, we get the following characterization of individual consumption

dynamics.

**Lemma 1.** *The evolution of individual consumption, investment, and wealth is given by:*

$$\frac{dc_t^i}{c_t^i} = \frac{dx_t^i}{x_t^i} = (\rho_t - \beta) dt + \sigma (1 - \tau_t) \phi_t dz_t^i, \quad (13)$$

where  $\rho_t = (1 - \tau_t) \phi_t r_t + (1 - \tau_t) (1 - \phi_t) R_t$  is the total return to saving. Solving for  $c_t^i$ , the evolution of individual consumption is:

$$c_t^i = c_0^i \cdot \exp\left\{ \int_0^t (\hat{\rho}_s - \beta) ds + \int_0^t \sigma (1 - \tau_s) \phi_s dz_s^i \right\}, \quad (14)$$

where  $\hat{\rho}_t = \rho_t - \frac{1}{2} \sigma^2 (1 - \tau_t)^2 \phi_t^2$  is the risk-adjusted return to saving.

The proof of equation (14) follows from Ito's lemma (see appendix for details). Here,  $\rho_t$  is the mean return to saving, namely the total or overall portfolio return for the household. In other words, the total return to saving is a weighted average of the (net-of-tax) marginal product of capital and the (net-of-tax) risk-free rate. Then, the risk-adjusted return to saving,  $\hat{\rho}_t$ , is the certainty equivalent of the overall portfolio return, and is lower than  $\rho_t$  because agents are risk averse and face risk in their consumption stream. It follows that, in equilibrium, it will have to be  $(1 - \tau_t) R_t < \hat{\rho}_t < \rho_t < (1 - \tau_t) r_t$ .

## 4.2 General equilibrium

The initial position of the economy is given by the cross-sectional distribution of  $(k_0^i, b_0^i)$  across households. Households choose plans  $\{c_t^i, l_t^i, k_t^i, b_t^i\}_{t \in [0, \infty)}$  for  $i \in [0, 1]$ , contingent on the history of their idiosyncratic shocks, and given the price sequence and the government policy, so as to maximize their lifetime utility. Idiosyncratic risk washes out in the aggregate. An equilibrium is then defined as a deterministic sequence of prices  $\{w_t, R_t, r_t\}_{t \in [0, \infty)}$ , policies  $\{\tau_t, T_t\}_{t \in [0, \infty)}$ , and macroeconomic variables  $\{C_t, K_t, Y_t, L_t, X_t\}_{t \in [0, \infty)}$ , along with a collection of individual contingent plans  $\{c_t^i, l_t^i, k_t^i, b_t^i\}_{t \in [0, \infty)}$  for  $i \in [0, 1]$ , such that the following conditions hold: (i) given the sequences of prices and policies, the plans are optimal for the households; (ii) the labor market clears,  $\int_i l_t^i = 1$ , in all  $t$ ; (iii) the bond market clears,  $\int_t b_t^i = 0$ , in all  $t$ ; (iv) the government budget constraint (4) is satisfied in all  $t$ ; and (v) the aggregates are consistent with individual behavior,  $C_t = \int_i c_t^i$ ,  $L_t = \int_i l_t^i = 1$ ,  $K_t = \int_i k_t^i$ ,  $Y_t = \int_i F(k_t^i, l_t^i) = F(\int_i k_t^i, 1)$  in all  $t$ . Note that the aggregates do not depend on the extent of wealth inequality, because individual policies are linear in wealth.

Define  $f(K) \equiv F(K, 1) = K^\alpha$ . From Proposition 1, the equilibrium ratio of capital to effective wealth and the equilibrium mean return to saving are identical across agents and can be expressed as functions of the capital stock and risk-free rate:  $\phi_t \equiv \phi(K_t, R_t)$  and  $\rho_t \equiv \rho(K_t, R_t)$ . Similarly, the wage is  $w_t \equiv w(K_t) = f(K_t) - f'(K_t)K_t = (1 - \alpha)f(K_t)$ . Using this, aggregating the policy rules across agents, and imposing bond market clearing, we arrive at the following characterization of the general equilibrium.

**Proposition 2.** *In equilibrium, the aggregate dynamics satisfy:*

$$\frac{\dot{X}_t}{X_t} = \rho_t - m_t \quad (15)$$

$$\dot{H}_t = (1 - \tau_t)R_t H_t - w_t - \tau_t f'(K) K_t \quad (16)$$

$$K_t = \frac{\phi_t}{1 - \phi_t} H_t, \quad (17)$$

along with  $m_t = \beta$  and  $\rho_t = (1 - \tau_t)\phi_t r_t + (1 - \tau_t)(1 - \phi_t)R_t$ .

The formal proof of Proposition 2 can be found in the appendix. Condition (15) follows from aggregating the individual wealth evolution constraints (12) across agents, and using the definition of  $\rho_t$ . It captures the evolution of total effective wealth, and shows that wealth grows when the mean return to saving,  $\rho_t$ , exceeds the marginal propensity to consume,  $\beta$ . Condition (16) is the evolution of human wealth from (6), combined with the aggregated government budget from (4). Condition (17) is obtained from aggregating over the policy functions in (10), using bond market clearing  $B_t = 0$ , and dividing. This equation represents clearing in the bond market and ensures that the bond is in zero net supply in the aggregate.

### 4.3 Steady state

This section characterizes the steady state and also shows the behavior of the variables in steady state, as a function of the capital tax.

#### 4.3.1 Steady-state aggregates

The steady state is the fixed point of the dynamic system described in Proposition 2. The following proposition characterizes the steady state.

**Proposition 3.** *(i) The steady state always exists and is unique. (ii) The steady state is the solution to the following system of three equations in the risky return,  $r$ , the risk-free interest*

rate,  $R$ , and the fraction of effective wealth invested in capital,  $\phi$ :

$$r\phi(1 - \tau) + R(1 - \phi)(1 - \tau) = \beta \quad (18)$$

$$r\phi = \alpha\beta \quad (19)$$

$$\phi = \frac{r - R}{\sigma^2(1 - \tau)}. \quad (20)$$

(iii) For any given value of the capital tax, the steady-state level of the capital stock,  $K$ , is below its corresponding level in complete markets.

The formal proof of Proposition 3 is in the appendix. Equation (18) combines stationarity of wealth from (15) with the definition of the mean return to saving,  $\rho$ . It says that wealth stationarity requires  $\rho = \beta$ , namely that the mean return to saving is equal to the marginal propensity to consume. But since  $(1 - \tau)R < \rho$ , it follows that  $(1 - \tau)R < \beta$  in steady state. This is a manifestation of the precautionary saving motive, with a rationale similar to that in Aiyagari (1994).<sup>6</sup> Equation (19) follows from the aggregate resource constraint of the economy, using factor market clearing and the aggregate policy functions. Equation (20) is just a re-write of portfolio allocation from (11).

For part (iii), note that in a model similar to the one in this paper, Angeletos (2007) shows that the level of steady-state capital will be below complete markets if and only if  $\theta > \phi/(2 - \phi)$ , where  $\theta$  is the elasticity of intertemporal substitution and  $\phi$  is the fraction of total effective wealth invested in capital.<sup>7</sup> This condition is always satisfied here, because  $\phi < 1$  always, and  $\theta = 1$  when preferences are logarithmic. Hence, in our model, the steady-state levels of capital, output and consumption will be lower than under complete markets (for the same level of the tax).

### 4.3.2 Steady-state aggregates and the capital tax

The following lemma characterizes the behavior of the aggregate variables with respect to the capital tax in steady state.

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<sup>6</sup>In particular, agents have a precautionary saving motive, because the idiosyncratic investment risk generates risk in their consumption stream. Therefore, if the net interest rate were higher than the discount rate in preferences, saving and wealth would explode, which violates the notion of steady state. In fact, the net interest rate has to be lower than the discount rate by exactly as much as is needed for the corresponding substitution effect of a lower saving return to exactly offset the precautionary saving motive.

<sup>7</sup>In general, Angeletos (2007) shows that steady-state capital may be either below or above complete markets, depending on whether the (private) risk-premium effect or the precautionary saving effect, respectively, dominates.

**Lemma 2.** (i) In steady state, the monotonicity of the aggregates with respect to the capital tax is described by the following:

$$\frac{d\phi}{d\tau} = \phi \frac{(r - R)(1 - 2\phi) - R}{(1 - \tau)[(r - R)(1 - 2\phi) + r]} \quad (21)$$

$$\frac{dr}{d\tau} = -\frac{\alpha\beta}{\phi^2} \frac{d\phi}{d\tau} \quad (22)$$

$$\frac{dR}{d\tau} = \frac{dr}{d\tau} - \frac{d\phi}{d\tau} \sigma^2 (1 - \tau) + \phi \sigma^2 \quad (23)$$

$$\frac{dH}{d\tau} = \frac{d\phi}{d\tau} \frac{K}{\phi^2} \frac{\alpha - \phi}{1 - \alpha} \quad (24)$$

$$\frac{dK}{d\tau} = \frac{1}{\alpha(\alpha - 1)} K^{2-\alpha} \frac{dr}{d\tau} \quad (25)$$

(ii) In steady state,  $dR/d\tau > 0$  necessarily. (iii) In steady state, it is possible that  $\phi$  is maximized for an interior value of the capital tax,  $\tau^* \in (-1, 1)$ .

The formal proof of lemma 2 can be found in the appendix. Here, note that equation (22) is immediate from (19), equation (23) follows from portfolio allocation in (20), and equation (25) comes from  $r = \alpha K^{\alpha-1}$ . To get equation (24), we combine (17), (22), and (25). The crucial derivative is then  $d\phi/d\tau$ , which is calculated from (18), (19), and (20), using the implicit function theorem. The appendix shows that the denominator of (21) is always positive, hence the sign of  $d\phi/d\tau$  depends on the sign of the numerator. In addition, if  $d\phi/d\tau = 0$  for  $\tau^* \in (-1, 1)$ , then  $\phi$  has an interior maximum at  $\tau = \tau^*$ . Then, from (25), which follows from  $r = \alpha K^{\alpha-1}$ , capital is maximized at  $\tau = \tau^*$ .

The possibility that, in this model, the capital stock might be maximized for an interior value of the tax, and hence might be increasing with the tax over some range of taxes, was established in Panousi (2012). There, it is shown that a general-equilibrium insurance aspect of the tax, operating through the increase in the interest rate, may lead to an increase in capital accumulation when the tax increases. Here, note additionally that, although  $K$  and  $\phi$  have the same monotonicity, the same need not be the case for human wealth. In particular, from (24) we can see that  $H$  may or may not have the same monotonicity as  $\phi$ , depending on the sign of the term  $\alpha - \phi$ . If  $K$ ,  $\phi$ , and  $H$  are all maximized for the same tax, then aggregate consumption  $C = \beta(K + H)$  will also be maximized at that tax. We will return to this point in section 5.4.

## 5 The ex ante optimal tax

The optimal tax in our framework is the one maximizing ex ante welfare. We will first characterize the planner's problem of maximizing ex ante welfare, and then we will evaluate the solution to the optimal taxation problem in the long run.<sup>8</sup>

### 5.1 Characterization of the planner's problem

The social planner's objective is to choose the tax sequence  $\{\tau_t\}_{t=0}^{\infty}$  that maximizes ex ante expected utility, subject to the conditions for individual optimization and general equilibrium in section 4. The planner's objective function,  $\tilde{W}$ , is then weighted sum of agents' value functions,  $V^i$ , from (9), where the weights,  $\psi(a_0^i)$ , depend on the initial financial wealth of each agent:

$$\tilde{W}(\{x_0^i\}; \{\tau_t\}_{t=0}^{\infty}) = \int_i V(x_0^i; \{\tau_t\}_{t=0}^{\infty}) \psi(a_0^i) da_0^i. \quad (26)$$

where  $x_0^i \equiv a_0^i + h_0(\{\tau_t\}_{t=0}^{\infty})$  and  $a_0^i = k_0^i + b_0^i$ . Without loss of generality, we will assume that at  $t = 0$  the wealth distribution is concentrated at one point, so that all agents hold the same amount of capital, equal to the economy-wide aggregate capital stock, and therefore receive the same weight in the planner's objective, so that  $a_0^i = a_0$  and  $\psi(a_0^i) = 1$ . The following proposition characterizes the planner's problem of maximizing ex ante welfare.

**Proposition 4.** *The planner chooses the ex ante optimal sequence of taxes  $\{\tau_t\}_{t=0}^{\infty}$  to maximize the objective function:*

$$\tilde{W}(x_0; \{\tau_t\}_{t=0}^{\infty}) = \int_0^{\infty} e^{-\beta t} \{ \ln(c_0) + \int_0^t (\hat{\rho}_t - \beta) ds \} dt \quad (27)$$

subject to the following constraints:

$$c_0 = \beta x_0 \quad (28)$$

$$x_0 = a_0 + h_0(\{\tau_t\}_{t=0}^{\infty}), \quad a_0 = k_0 + b_0 \text{ given} \quad (29)$$

$$\hat{\rho}_t = (1 - \tau_t)r_t\phi_t + (1 - \tau_t)R_t(1 - \phi_t) - \frac{1}{2}\sigma^2(1 - \tau_t)^2\phi_t^2 \quad (30)$$

$$\phi_t = \frac{r_t - R_t}{\sigma^2(1 - \tau_t)}, \quad (31)$$

$$dK_t = [K_t^\alpha - \beta X_t]dt, \quad (32)$$

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<sup>8</sup>Note, however, that we are taking the transition to the long-run optimal steady state explicitly into account, and that we are not imposing a constant tax along the transition.

$$dX_t/X_t = [(1 - \tau_t)r_t\phi_t + (1 - \tau_t)R_t(1 - \phi_t) - \beta]dt , \quad (33)$$

$$r_t = \alpha K_t^{\alpha-1}, \quad w_t = (1 - \alpha)K_t^\alpha , \quad (34)$$

$$T_t = \tau_t r_t K_t . \quad (35)$$

It then follows that, at time  $t = 0$ , the first order condition with respect to a change in the capital tax in period  $t = u$  is:

$$\frac{d\tilde{W}}{d\tau_u} = \frac{\partial\tilde{W}}{\partial x_0} \frac{dx_0}{d\tau_u} + \frac{\partial\tilde{W}}{\partial\phi_u} \frac{d\phi}{d\tau_u} + \frac{\partial\tilde{W}}{\partial r_u} \frac{dr}{d\tau_u} + \frac{\partial\tilde{W}}{\partial R_u} \frac{dR}{d\tau_u} + \frac{\partial\tilde{W}}{\partial\tau_u} = 0 . \quad (36)$$

In other words, (36) is the planner's first order condition that defines the optimal tax.

The complete proof of Proposition 4 can be found in the appendix. The planner's objective in (27) obtains from using the agents' value functions in (9) and the law of motion for agents' consumption in (14), together with the fact that all uncertainty is idiosyncratic and hence  $E_i \int_0^t \sigma(1 - \tau_s)\phi_s dz_s^i = 0$ . Equation (28) is the consumption policy function. Equation (29) is the definition of initial effective wealth, which is the sum of the historically given financial wealth,  $a_0^i \equiv k_0^i + b_0^i$ , and of human wealth,  $h_0$ . The latter depends on the entire future sequence of wages and prices, as we discuss below. Equations (30) and (31) simply repeat the definition of the risk-adjusted return to saving and the determination of optimal portfolio allocation, respectively. Equation (32) is the aggregate resource constraint in the economy. Equation (33) is the aggregate wealth evolution constraint from (15), using the definition of the mean return to saving,  $\rho_t$ . Equation (34) is market clearing for the factors of production. Equation (35) is the aggregate government budget constraint.

## 5.2 Discussion of the planner's problem

The planner maximizes the objective function (27), which consists of two terms. The first term captures the effect of the entire path of future prices and tax policies on consumption at time zero,  $c_0$ . This is because consumption depends on effective wealth,  $x_0$ , which is the sum of the historically given asset holdings,  $a_0 = k_0 + b_0$ , and of human wealth,  $h_0$ , where human wealth is the present discounted value of future wages and transfers (namely of future safe income). Therefore,  $h_0$ , and thereby  $c_0$ , depends on the entire future path for the tax, the wage, the return to capital, and the interest rate.

The second term in the planner's objective captures direct or partial equilibrium and indirect or general equilibrium welfare effects of the tax. The direct welfare effects can be seen immediately from equation (30). Specifically, the tax directly reduces the mean return

to saving (role of  $(1 - \tau)$  in the first two terms of (30), holding  $r_t$  and  $\phi_t$  constant), but it also reduces the effective volatility of risk (role of  $(1 - \tau)$  in the third term of (30), holding  $\phi_t$  constant). The indirect welfare effects of the tax then operate through allowing for the general equilibrium adjustment of the risky return,  $r_t$ , the risk-free return,  $R_t$ , and the portfolio allocation,  $\phi_t$ , when the tax changes. We will come back to these effects of the tax in the following sections.

Therefore, the planner has to weigh two considerations when choosing the optimal tax sequence. First, how the path of the taxes will affect the paths of wages and prices, thereby influencing time-zero consumption,  $c_0$ , through  $h_0$ . Second, how the path of the taxes will affect the difference between the risk-adjusted return to saving,  $\hat{\rho}_t$ , and the marginal propensity to consume,  $\beta$ , namely the risk-adjusted rate of growth of household consumption and wealth (see equation (14)).

In addition, if  $x_0$  did not include human wealth, it would be historically given and therefore irrelevant for the planner's maximization problem. In turn, absence of human wealth means that there is only risky income in the economy, as in an  $AK$  version of our model (without transfers). In that case, the planner chooses the path of taxes to maximize  $\hat{\rho}_t - \beta$ , which is exactly the difference between the risk-adjusted return to saving and the marginal propensity to consume, or the risk-adjusted growth rate of individual consumption and wealth.

### 5.3 The planner's first order condition

The planner's first order condition (36) characterizes the optimal tax. We will now proceed in three steps to calculate all the derivatives entering (36). First, we will estimate the various derivatives of the planner's objective function,  $\tilde{W}$ . Second, we will calculate the derivative of initial effective wealth,  $x_0$ , and therefore of initial human wealth,  $h_0$ , with respect to the tax. Third, we will find the derivatives of portfolio allocation,  $\phi$ , and of asset returns,  $r$  and  $R$ , with respect to the tax.

To begin with, the following lemma characterizes the derivatives of the planner's objective function.

**Lemma 3.** *The derivatives of the planner's objective function,  $\tilde{W}$ , entering the first order*

condition (36) are given by:

$$\frac{\partial \tilde{W}}{\partial x_0} = \frac{1}{x_0} \int_0^\infty e^{-\beta s} ds \quad (37)$$

$$\frac{\partial \tilde{W}}{\partial \phi_u} = [(1 - \tau_u)r_u - (1 - \tau_u)R_u - \sigma^2(1 - \tau_u)^2\phi_u] \int_u^\infty e^{-\beta s} ds = 0 \quad (38)$$

$$\frac{\partial \tilde{W}}{\partial r_u} = (1 - \tau_u)\phi_u \int_u^\infty e^{-\beta s} ds \quad (39)$$

$$\frac{\partial \tilde{W}}{\partial R_u} = (1 - \tau_u)(1 - \phi_u) \int_u^\infty e^{-\beta s} ds \quad (40)$$

$$\frac{\partial \tilde{W}}{\partial \tau_u} = [-r_u\phi_u - R_u(1 - \phi_u) + \sigma^2(1 - \tau_u)\phi_u^2] \int_u^\infty e^{-\beta s} ds = -R_u \int_u^\infty e^{-\beta s} ds, \quad (41)$$

where the subscript  $u$  denotes the variable at time  $t = u$ , and where  $\int_u^\infty e^{-\beta s} ds = e^{-\beta u}/\beta$ .

Simple differentiation of  $\tilde{W}$  yields equation (37). The formal derivation of  $\partial \tilde{W}/\partial \phi_u$ ,  $\partial \tilde{W}/\partial r_u$ ,  $\partial \tilde{W}/\partial R_u$ , and  $\partial \tilde{W}/\partial \tau_u$  is more convoluted, and is left for the appendix. Intuitively, these derivatives indicate the effect on the objective at time  $t = 0$  of a change in the functions  $\phi_t$ ,  $r_t$ , and  $R_t$  at time  $t = u$ , due to a change in the tax at that point in time. Hence, the proofs in the appendix rely on the use of functional derivatives, with the Dirac delta function as the appropriate test function. In turn, the Dirac delta function allows for an impulse change in the functions at time  $t = u$ , due to a change in the tax at that time, while the functions remain unchanged at all other points in time.

The derivative in (38) captures indirect or general equilibrium effects of the tax on welfare operating through portfolio allocation. Using (11), we get that this derivative is actually zero. This is because the optimal choice of  $\phi$  actually maximizes the risk-adjusted return to saving,  $\hat{\rho}$ , as can be seen from (30). Equations (39) and (40) capture indirect or general equilibrium effects of the tax on welfare operating through asset returns, for the risky (capital) asset and the riskless (bond) asset, respectively. The weights on these terms, which depend on  $\phi$ , capture the importance of each asset on the overall portfolio. Equation (41) reflects the direct or partial equilibrium effect of the tax on welfare, and consists of two terms. The first term,  $-[r\phi + R(1 - \phi)]$ , is negative, reflecting the standard effect that the tax lowers the mean return to saving. The second term,  $\sigma^2(1 - \tau)\phi$ , is positive, reflecting the fact that the tax directly provides some insurance by lowering the effective variance of risk,  $\sigma(1 - \tau)$ . Using (11), we get that the partial equilibrium term in brackets in (41) equals  $-R$ , and hence is always negative. This means that the distortionary effect of the tax on saving outweighs its direct insurance effect.

Second, the following lemma characterizes the derivative  $dx_0/d\tau_u$  entering the planner's first order condition (36).

**Lemma 4.** (i) *Total effective wealth at  $t = 0$  is given by  $x_0 = a_0 + \int_0^\infty e^{-\int_0^s p_j dj} q_s ds$ , where  $a_0 \equiv k_0 + b_0$  is historically given, and where  $p_t$  and  $q_t$  are, respectively, the equilibrium after-tax interest rate and the equilibrium safe income from wages and transfers as a function of prices only:*

$$p_t \equiv (1 - \tau_t)R_t, \quad (42)$$

$$q_t \equiv w_t + T_t = [(1 - \alpha)\alpha^{\alpha/(1-\alpha)} + \tau_t\alpha^{1/(1-\alpha)}]r_t^{\alpha/(\alpha-1)}. \quad (43)$$

(ii) *The derivative of  $x_0$ , with respect to a change in the capital tax at time  $t = u$ , is:*

$$\frac{dx_0}{d\tau_u} = \frac{dh_0}{d\tau_u} = e^{-\int_0^u p_s ds} \frac{dq}{d\tau_u} - \frac{dp}{d\tau_u} \int_u^\infty q_t e^{-\int_0^t p_s ds} dt, \quad \text{where} \quad (44)$$

$$\frac{dp}{d\tau_u} = -R_u + (1 - \tau_u) \frac{dR}{d\tau_u}, \quad (45)$$

$$\frac{dq}{d\tau_u} = \alpha^{1/(1-\alpha)} r_u^{\alpha/(\alpha-1)} + [(1 - \alpha)\alpha^{\alpha/(1-\alpha)} + \tau_u\alpha^{1/(1-\alpha)}] \alpha(\alpha - 1)^{-1} r_u^{1/(\alpha-1)} \frac{dr}{d\tau_u}. \quad (46)$$

where  $dr/d\tau_u$  and  $dR/d\tau_u$  are the derivatives of the risky and riskless asset with respect to the tax at time  $t = u$ .

The formal proof of Lemma 4 is left for the appendix. The proof uses the definition of a functional derivative, the Dirac delta function, and the series expansion for the exponential. It also employs the fact that the wage is given by  $w_t = (1 - \alpha)(r_t/\alpha)^{\alpha/(\alpha-1)}$  and the transfers are given by  $T_t = \tau_t r_t^{\alpha/(\alpha-1)} \alpha^{1/(1-\alpha)}$ , using equations (34) and (35). Though we have been unable to explicitly characterize the derivatives  $d\phi/d\tau_u$ ,  $dr/d\tau_u$ , and  $dR/d\tau_u$  along the transition, Lemma 2 has characterized these derivatives in steady state. Therefore, in the next section, we will proceed to evaluate the optimal tax in steady state.

## 5.4 Evaluating the ex ante optimal tax in steady state

We can now substitute the derivatives from Lemmas 2, 3, and 4 into the the planner's first order condition (36), and evaluate that condition in the long run, as  $u \rightarrow +\infty$ . In other words, we will proceed to find the long-run or steady-state value of the ex ante optimal tax. Though we evaluate the optimal tax in steady state, it is important to clarify that we have nonetheless explicitly taken into account the transitional dynamics and that we have not

imposed a constant tax along the transition.<sup>9</sup> Using the notation  $y_\tau$  to denote the derivative of a variable  $y$  with respect to the tax and dropping the dependence of the variables on the tax for notational simplicity, the following theorem characterizes the long-run value of the optimal tax.

**Proposition 5.** *(i) In the long run, the planner's first order condition is:*

$$\lim_{u \rightarrow \infty} \left\{ (q_\tau - p_\tau \frac{q}{p}) + x_0 [(1 - \tau)\phi r_\tau + (1 - \tau)(1 - \phi)R_\tau - R] \frac{e^{-\beta u}}{e^{-pu}} \right\} = 0 \quad (47)$$

where  $p$ ,  $q$ ,  $p_\tau$ , and  $q_\tau$  are given by (42), (43), (45), and (46), respectively, in steady state; where  $r_\tau$  and  $R_\tau$  are given by (22) and (23), respectively; and where the second term goes to zero as  $u \rightarrow \infty$ , because  $p = (1 - \tau)R < \beta$ .

*(ii) Hence, the ex ante optimal tax, evaluated at its long-run steady state, solves:*

$$\frac{dH}{d\tau} = 0 \Leftrightarrow \frac{dq}{d\tau}p - \frac{dp}{d\tau}q = 0, \quad (48)$$

where  $H = q/p$  is the steady-state value of human wealth. In other words, the optimal tax in the long run always maximizes steady-state human wealth, namely the present value of wages and transfers,  $q$ , discounted by the after-tax riskless rate,  $p$ .

*(iii) Using  $H_\tau = \phi_\tau(\alpha - \phi)K\phi^{-2}(1 - \alpha)^{-1}$  from (24), it can be shown that there are two possible cases for the optimal steady state. (a)  $\phi = \alpha$ : Here, capital is at its first best level. The optimal tax is always negative and it does not maximize aggregate consumption. (b)  $d\phi/d\tau = 0$ : Here, capital is at its highest possible level under incomplete markets. The optimal tax can be either positive or negative, and it also maximizes aggregate consumption.*

The formal proof of Proposition 5 is left for the appendix. Part (i) follows from substituting the derivatives in Lemmas 2, 3, and 4 into the planner's first order condition (36), and evaluating this condition in steady state. Part (ii) states that, in the long run, the ex ante optimal tax always maximizes steady-state human wealth,  $H$ , namely the present discounted value of safe wage and transfer income.<sup>10</sup> Alternatively, the ex ante optimal tax in the long run always solves equation (48). Part (iii) then uses equation (24) to substitute for the derivative of human wealth with respect to the tax in (48). Clearly then, there are two possible cases for the optimal steady state.

<sup>9</sup>In other words, we study the ex ante optimal tax evaluated at steady state, and not the steady-state optimal tax (we do not perform any steady-state welfare exercises in this paper).

<sup>10</sup>Essentially, at the optimal steady state, the planner places all the weight on the first term of his objective function (27).

In the first scenario, described in (iii) – (a), maximizing human wealth entails  $\phi = \alpha$ , which, as the appendix shows, actually characterizes the first-best allocation. By first best we mean the allocation that would obtain in our model for  $\sigma^2 = 0$ , or in other words, the allocation of the corresponding complete-markets economy that also allows for lump-sum transfers. So, in this case, the economy-wide aggregates in our model behave with the tax as though markets were complete. Hence, capital is monotonically falling with the tax, and the optimal tax is a subsidy. In fact, this optimal subsidy increases the steady-state capital stock all the way to its corresponding first-best level. Furthermore, as with complete markets, the ex ante optimal tax does not maximize long-run consumption.<sup>11</sup>

In the second scenario, maximizing human wealth entails  $d\phi/d\tau = 0$ . This means that the aggregates are at an interior optimum with respect to the tax, as explained in Lemma 2. Therefore, in this case, the optimal tax maximizes not only human wealth, but also aggregate capital and consumption, from Lemma 2 and from the fact that  $C = \beta(K + H)$ . This is in contrast to the case of complete markets, where the ex ante optimal tax does not maximize steady-state consumption.<sup>12</sup> Recall from Proposition 3 that, in our model, steady-state capital is below its first-best level. This means that the optimal tax here, by maximizing capital, brings the steady-state capital stock as close as possible to the first best, given the constraints of the model. However, the sign of the optimal tax in this case turns out to be ambiguous: It can be either negative, if the maximum capital stock is attained in the negative region for taxes, or positive, if the maximum capital stock is attained in the positive region for taxes.

Overall then, the optimal tax, evaluated at steady state, always maximizes human wealth. Furthermore, although it cannot be unambiguously signed, the optimal tax is always chosen so as to increase the capital stock to its maximum possible level, given the economy’s constraints. In the case where capital is falling with the tax, the optimal tax is negative, and it actually increases capital all the way to its first-best level. In the case where capital is inversely-U shaped with the tax, the optimal tax is such that capital is at its maximum level under incomplete markets, so that the optimal tax again brings capital as close as possible to the first best.

The rationale for optimal taxation in our model has a similar flavor as that in Aiyagari (1995). There, the uninsurable labor-income shocks induce precautionary saving, which

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<sup>11</sup>In complete markets, the *ex ante* optimal tax is zero, and it does not maximize long-run consumption. The *steady-state* optimal tax is actually a subsidy, and it maximizes consumption by taking the economy to the golden rule.

<sup>12</sup>Here, the tax maximizes steady-state consumption, although optimal consumption is still below complete markets, because of the presence of idiosyncratic risk. See Angeletos (2007) for the effect of such risk on the aggregates.

leads to overaccumulation of capital in steady state, compared to the first best. In addition, capital always falls with the tax. Hence, the optimal tax is positive so as to bring capital to its first-best level. In our model, the optimal capital tax also performs the role of bringing steady-state capital as close to the first best as possible.

However, the mechanism in our model is different from Aiyagari (1995) in two meaningful ways: In our framework, steady-state capital may be below complete markets, and capital may be increasing with the tax. First, the uninsurable investment risk may lead to underaccumulation of capital, compared to the first best, if the negative effect on investment induced by the risk premium agents require to invest in capital dominates the standard precautionary saving effect.<sup>13</sup> For empirically relevant parameter values, including the log utility that is considered in our baseline model, our steady-state capital stock is then below complete markets, whereas Aiyagari's is above complete markets. Hence, in our model, the optimal tax aims to increase investment, whereas in Aiyagari it aims to reduce investment. Second, the way capital accumulation reacts to capital taxation in our model is no longer monotone, as in Aiyagari, since a higher tax can increase capital investment due to general-equilibrium effects operating through the interest-rate adjustment.<sup>14</sup> Hence, the optimal tax cannot be unambiguously signed.

However, the sign of the optimal tax can be determined by reference to the degree of market incompleteness. Intuitively, for low levels of idiosyncratic risk, the aggregates respond to the tax as if markets were complete, capital is decreasing with the tax, and the optimal tax is a subsidy, since a subsidy induces capital accumulation. For high levels of risk, capital is increasing with the tax, and the optimal tax is positive, since a positive tax increases capital toward the first best. The following lemma demonstrates how the optimal steady state as well as the sign of the optimal tax depend on the exogenous level of idiosyncratic risk in the economy.

**Lemma 5.** *(i) If the volatility of idiosyncratic risk is below a minimum lower bound, namely  $0 < \sigma < \underline{\sigma} \equiv \sqrt{\beta}$ , then the optimum is described by  $\phi = \alpha$  and the optimal tax is always negative. If the volatility of idiosyncratic risk is above this lower bound, namely  $\sigma > \underline{\sigma}$ , then the optimum is described by  $d\phi/d\tau = 0$  and the optimal tax can be either positive or negative. (ii) If the volatility of risk is sufficiently high, namely  $\sigma > \bar{\sigma} \equiv \sqrt{4\beta/\alpha(2-\alpha)}$ , then the optimal tax is always positive.*

The proof of Lemma 5 can be found in the appendix. This lemma basically shows that the risk threshold determining if the optimal tax is described by (iii) – (a) or (iii) – (b) in

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<sup>13</sup>See Angeletos (2007) for the original result.

<sup>14</sup>See Panousi (2012) for the original result.

Proposition 5 is  $\underline{\sigma} \equiv \beta$ , while the risk threshold determining if the optimal tax is positive or negative is  $\bar{\sigma} \equiv 4\beta/\alpha(\alpha - 2)$ . Part (i) states that, when the volatility of idiosyncratic risk is very low, in particular below a minimum lower bound,  $\underline{\sigma} = \sqrt{\beta}$ , then the economy is very close to complete markets. Hence, capital is falling with the tax and the optimal tax is a subsidy. When the volatility of risk is above this lower bound  $\sigma > \underline{\sigma} = \sqrt{\beta}$ , then the general equilibrium mechanisms identified in Panousi (2012) are strong enough to lead to a steady-state capital stock that is inversely U-shaped with the tax. In this case, the first best is not attainable, and the optimal tax, which can be either positive or negative, will be such that it brings the capital stock to its maximum, namely as close as possible to its level under complete markets. Part (ii) then shows that, if risk is sufficiently enough, specifically if  $\sigma > \bar{\sigma} = \sqrt{4\beta/\alpha(2 - \alpha)}$ , then the optimal tax is always positive.<sup>15</sup>

Therefore, the specific mechanism through which the optimal tax performs its role of bringing capital as close as possible to the first best depends on the level of uninsurable idiosyncratic risk in the economy. When risk is very low, risk-taking is encouraged through the optimal capital subsidy, which increases the mean return to saving in partial-equilibrium. When risk is sufficiently high, risk-taking is encouraged through a positive optimal tax, which, though reducing the mean return to saving, nonetheless induces general equilibrium insurance effects (documented in Panousi (2012)) strong enough to lead to increased capital accumulation.

## 5.5 Graphical representation of the optimal tax

For comparison purposes, Figure 1 presents the case of complete markets, where the volatility of idiosyncratic risk is  $\sigma = 0$ . The other parameter values are  $\alpha = 0.5$  and  $\beta = 0.04$ . Panels (a), (b), and (c) present the steady-state values for aggregate capital, human wealth, and aggregate consumption as a function of the capital tax, which ranges in  $[-1, 1]$ . Panel (d) graphs the planner's first order condition (48) as a function of the tax. Panel (a) shows that steady-state capital is monotonically decreasing with the tax. Panel (b) shows that human wealth is maximized when the tax is  $\tau^* = 0$ . This is the optimal tax under complete markets, as shown also in panel (d), where the line representing the first order condition crosses zero at  $\tau^* = 0$ . This shows that, under complete markets, the optimal tax is zero, and that it maximizes human wealth, but not the aggregate capital stock or aggregate consumption.

Figure 2 presents the case where risk is positive but below the minimum lower bound  $\underline{\sigma} = \sqrt{\beta}$ , so that  $0 < \sigma^2 < \beta$ . The parameter values have been set at  $\sigma^2 = 0.02$ ,  $\alpha = 0.5$  and

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<sup>15</sup>Note that an increase in patience or in the share of capital in production lower the threshold  $\bar{\sigma}$ , and hence make it more likely that the optimal tax will be positive.

$\beta = 0.04$ . Panel (a) shows that here, as in complete markets, capital is always decreasing with the tax. Panel (b) shows that the tax maximizing human wealth is  $\tau^* = -0.17$ . This is the optimal subsidy, as shown also in panel (d), where the first order condition crosses zero at  $\tau^* = -0.17$ . However, this subsidy does not maximize aggregate capital or consumption.

Figure 3 presents the borderline case where  $\sigma = \underline{\sigma} = \sqrt{\beta}$ . The parameter values have been set at  $\sigma^2 = 0.04$ ,  $\alpha = 0.5$  and  $\beta = 0.04$ . Panel (a) shows that capital is falling monotonically with the tax. In panel (d), the first order condition crosses the horizontal axis at  $\tau^* = 1 - 1/\alpha = -1$ , which is the lowest possible tax (maximum allowable subsidy). This is also the value of the tax that maximizes human wealth, as well as aggregate capital and consumption.

Figure 4 presents the case where risk is above the minimum lower bound,  $\underline{\sigma}$ , but below the upper bound,  $\bar{\sigma}$ , so that  $\beta < \sigma^2 < 4\beta/\alpha(2 - \alpha)$ . The parameter values have been set at  $\sigma^2 = 0.12$ ,  $\alpha = 0.5$  and  $\beta = 0.04$ . Panel (a) shows that capital is now inversely-U-shaped with the tax. In this case, we have seen that the optimal tax could be either positive or negative. Panels (b) and (d) show that the optimal tax, which maximizes human wealth and at which the first order condition crosses zero, here happens to be  $\tau^* = -0.25 < 0$ . The optimal tax also maximizes aggregate capital and consumption.

Figure 5 shows the borderline case where  $\sigma = \bar{\sigma}$ , so that  $\sigma^2 = 4\beta/\alpha(2 - \alpha)$ . The parameter values have been set at  $\sigma^2 = 0.2$ ,  $\alpha = 0.5$  and  $\beta = 0.04$ . The first order condition in panel (d) crosses the horizontal axis from above at the optimal tax  $\tau^* = 0$ . This tax maximizes human wealth, as well as aggregate capital and consumption. Note that, although the optimal tax here is the same as under complete markets, steady-state capital and consumption are actually lower, because of the existence of idiosyncratic risk.<sup>16</sup>

Figure 6 shows the case where risk is sufficiently high or  $\sigma = \bar{\sigma}$ , so that  $\sigma^2 > 4\beta/\alpha(2 - \alpha)$ . The parameter values have been set at  $\sigma^2 = 0.4$ ,  $\alpha = 0.5$  and  $\beta = 0.04$ . In this case, the optimal tax will always be positive. Here, the first order condition in panel (d) crosses the horizontal axis from above at the optimal tax  $\tau^* = 22$ . This tax maximizes human wealth, as well as aggregate capital and consumption.

In summary then, the optimal tax always maximizes human wealth. When risk is very low (region below  $\underline{\sigma}$ ), then, as in the case of complete markets, ex ante the optimal tax does not maximize aggregate capital or consumption. When risk is above a minimum lower bound (region above  $\underline{\sigma}$ ), then the optimal tax maximizes not only human wealth, but also aggregate capital and consumption. This optimal tax will still be negative for intermediate values of risk, but it will become positive for sufficiently high values of risk (region above  $\bar{\sigma}$ ).

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<sup>16</sup>See Angeletos (2007) for the effect of idiosyncratic investment risk on the steady-state aggregates.

Figure 7 summarizes the relationship between the level of risk and the optimal tax, for parameter values  $\alpha = 0.5$ ,  $\beta = 0.04$ , and  $\sigma \in [0, 1]$ . Panel (a) plots the optimal tax, evaluated at steady state, as a function of risk,  $\sigma$ . The optimal tax is zero when  $\sigma = 0$  (complete markets), and it becomes an increasing subsidy as risk increases up until  $\sigma = \sqrt{\beta} = 0.2$ . From that point on, the optimal tax is increasing with risk, and it becomes positive when  $\sigma = \sqrt{4\beta/\alpha(2-\alpha)} = 0.46$ . Panel (b) plots human wealth and aggregate capital, as a function of risk. For low risk,  $\sigma < \sqrt{\beta}$ , both  $H$  and  $K$  are at their first best levels. For risk above the minimum threshold,  $H$  and  $K$  are both falling with risk. Panel (d) shows that aggregate consumption follows a similar pattern with respect to risk. Panel (c) shows the behavior of various saving returns as risk increases. These patterns follow from the behavior of the optimal tax in panel (a), combined with Lemma 2.

## 6 Optimal tax along the transition

In this section we perform the following exercise. Assume that the economy initially rests at the long run optimal steady state, described in Proposition 5, with optimal tax  $\tau^*$ . Then, at time  $t = t_0$ , the planner considers a deviation from the optimal tax to a tax  $\tau_{t_0}$ , because such a deviation might be welfare-increasing from the point of view of agents at time  $t_0$ . The relevant question then is whether  $\tau_{t_0} > \tau^*$  or  $\tau_{t_0} < \tau^*$ . In other words, will a deviation at time  $t = t_0$  entail a higher or a lower optimal tax than  $\tau^*$ ?

The answer to this question basically depends on the sign of the second term in equation (47). This term captures the way the economy transitions toward the long run. The appendix shows that this term is positive, leading to the following proposition.

**Proposition 6.** *The ex ante optimal tax is higher along the transition than in the long-run steady state.*

Proposition 6 therefore states that, from the point of view of agents at time  $t_0$ , a deviation to a tax  $\tau_{t_0} > \tau^*$  is optimal. In other words, the ex ante optimal tax during the transition toward the long-run equilibrium is higher than the ex ante optimal tax in the long run. Recall from equations (27) and (38) – (40) that  $(1 - \tau)\phi r_\tau + (1 - \tau)(1 - \phi)R_\tau$  in the second term of (47) represents the general equilibrium effect of the tax on welfare, namely the effect of the tax on welfare through the adjustment of asset returns and portfolio allocation. Also, recall from (41), that  $-R$  in this term captures the partial equilibrium effect of the tax on welfare, namely the fact that the tax directly reduces the mean return to saving and the effective variance of risk (where the distortionary effect always dominates). The fact that

this entire term is positive means that, along the transition to the steady state, the general-equilibrium effects of the tax on asset returns (identified in Panousi (2012)) outweigh the partial-equilibrium distortionary effect of the tax on the mean return to saving. As a result, the ex ante optimal tax is higher along the transition than in the long-run steady state.

## 7 Conclusions

Our paper is the first in the general-equilibrium Ramsey literature of heterogeneous agents and incomplete markets to provide an analytic characterization of the optimal tax in the long run and along the transition to the steady state. First, the ex ante optimal tax, evaluated in the long run, always maximizes human wealth, namely the present discounted value of agents' income from sources that are not subject to risk. The sign of the optimal tax depends on the strength of distortion versus insurance or redistribution considerations. When idiosyncratic risk in the economy is below a minimum lower bound, the economy is sufficiently "close" to its complete-markets counterpart and redistribution considerations are relatively weak. In this case, the optimal tax is negative, brings capital to its first-best level, and does not maximize aggregate consumption. When risk is above this lower bound, markets are sufficiently incomplete and redistribution considerations are relatively strong. In this case, the optimal tax can be either positive or negative, brings the capital stock to its maximum feasible level given the economy's constraints, and maximizes aggregate consumption. Furthermore, when risk is above a certain upper bound, then the optimal tax is necessarily positive. In all cases, the optimal tax encourages risk taking and increases capital accumulation. Second, we show analytically that the transition to the long-run optimal tax occurs monotonically from above, so that the short run entails higher capital taxes than the long run. This is essentially due to general-equilibrium effects of the tax on asset returns, which outweigh the partial-equilibrium distortionary effect of the tax on the mean return to saving.

In sum, we find that the the rationale for positive capital-income taxation in the long run does not necessarily extend to the case where markets are incomplete due to the presence of idiosyncratic capital-income risk. In particular, the sign of the optimal tax will depend on the extent of market incompleteness. In addition, our analysis suggests the source of market incompleteness, for example whether it is due to uninsurable idiosyncratic labor- or capital-income risk, as well as the potential correlation between these two sources of risk, will be crucial for determining the sign and the quantitative magnitude of the optimal capital tax.

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## Appendix

**Proof of Proposition 1. Utility maximization.** Because of the CRRA/CEIS specification of preferences, guess that the value function is:

$$U(x, t) = \frac{1}{\beta} \log(x),$$

The Bellman equation is:

$$\begin{aligned} 0 = \max_{c, \phi} \log(c, U(x, t)) + \frac{\partial U}{\partial x}(x, t)[(\phi(1 - \tau_t)r_t + (1 - \phi)(1 - \tau_t)R_t)x - c] \\ + \frac{\partial U}{\partial t}(x, t) + \frac{1}{2} \frac{\partial^2 U}{\partial x^2}(x, t)\sigma^2(1 - \tau_t)^2\phi^2x^2, \end{aligned}$$

The first order condition for the optimal portfolio allocation gives the condition for  $\phi_t$  in the main text. The first order condition for consumption gives  $c = \beta x$

**Proof of Lemma 1. Individual laws of motion.** Equation (13) follows from equation (12), using the definition of  $\rho_t$ . Applying Ito's Lemma to equation (13) and using the definition of  $\hat{\rho}_t$  yields equation (14).

**Proof of Proposition 2. General equilibrium.** Let  $p_t$  be the after-tax risk-free rate, so that  $p_t \equiv (1 - \tau_t)R_t$ . The human wealth for each individual in the economy is  $h_t^i = \int_t^\infty e^{-\int_t^s p_j dj} (w_s + T_s) ds$ . Aggregate human wealth is  $H_t = \int_t^\infty e^{-\int_t^s p_j dj} (w_s + T_s) ds = h_t^i$ . Using the Leibniz rule, and substituting from the government budget constraint, we get that the evolution of total human wealth is described by equation (16) in the main text. The aggregate capital stock in the economy is given by  $K_t = \phi_t X_t$ . Since  $B_t = 0$  in the aggregate economy, aggregating over the individual policy functions for the bond yields  $H_t = (1 - \phi_t)X_t$ . Adding up these last two equations gives  $X_t = K_t + H_t$ . Dividing these equations gives (17) in text. Aggregating over the individual wealth evolution constraints in (12) and using the definition of  $\rho_t$ , we get equation (15) in text.

**Proof of Proposition 3. Steady state.** (i) For the proof of existence and uniqueness of the steady state, See Panousi (2012). (iii) For the steady state comparison to complete markets, see Angeletos (2007). (ii) Equation (20) is simply a re-write of the portfolio allocation condition (11) in steady state. Aggregating over the individual consumption functions we get that  $C_t = \beta X_t$ . Using this, and since  $\delta = G = 0$ , we get that the aggregate resource constraint in the economy is:

$$dK_t = [K_t^\alpha - \beta X_t] dt .$$

Using  $f(K_t) = K_t^\alpha$  and  $r_t = f'(K_t)$ , we get that  $K_t^\alpha = r_t K_t / \alpha$ . Using this, as well as the aggregate policy functions  $K_t = \phi_t X_t$  and  $C_t = \beta X_t$ , we can then write the resource constraint as:

$$dK_t = [r_t \phi_t / \alpha - \beta] X_t dt .$$

In steady state, where  $dK_t = 0$ , the aggregate resource constraint yields:

$$r\phi = \alpha\beta ,$$

which is equation (19) in text. Aggregating over the individual wealth accumulation constraints in (12) and using the aggregate policy functions, gives:

$$dX_t = [(1 - \tau_t)r_t\phi_t + (1 - \tau_t)R_t(1 - \phi_t) - \beta]X_t dt .$$

In steady state, where  $dX_t = 0$ , this gives:

$$r\phi(1 - \tau) + R(1 - \tau)(1 - \phi) = \beta ,$$

which is equation (18) in text.

**Proof of Lemma 2. Steady state aggregates and the capital tax.** Conditions (18), (19), and (20) are a system of three equations in three unknowns,  $r$ ,  $R$ , and  $\phi$ . In turn, the three unknowns will be a function of the capital tax, so that  $r \equiv \mathbf{r}(\tau)$ ,  $R \equiv \mathbf{R}(\tau)$ , and  $\phi \equiv \phi(\tau)$ , where the bold letters indicate functions. We can then write these equations as:

$$M(\phi(\tau)) \equiv (1 - \tau)\left(\frac{\alpha\beta}{\phi(\tau)} - \phi(\tau)\sigma^2(1 - \tau)\right)(1 - \phi(\tau)) - \beta[1 - (1 - \tau)\alpha] = 0 . \quad (18')$$

$$\mathbf{r}(\tau) = \frac{\alpha\beta}{\phi(\tau)} , \quad (19')$$

$$\mathbf{R}(\tau) = \mathbf{r}(\tau) - \phi(\tau)\sigma^2(1 - \tau) , \quad (20')$$

One can then solve (18') to find portfolio allocation as a function of the tax,  $\phi = \phi(\tau)$ . Furthermore, the implicit function theorem, applied to (18'), can be used to determine if  $\phi$ , and consequently (as shown below) the economy-wide aggregates, are maximized at an interior value of the tax. In particular:

$$\frac{d\phi}{d\tau} = -\frac{\partial M/\partial\tau}{\partial M/\partial\phi} . \quad (IFT)$$

Simple differentiation of (18') gives:

$$\frac{\partial M}{\partial\tau} = -\left(\frac{\beta\alpha}{\phi} - \phi\sigma^2(1 - \tau)\right)(1 - \phi) + (1 - \tau)(1 - \phi)\phi\sigma^2 - \alpha\beta ,$$

and also that:

$$\frac{\partial M}{\partial\phi} = (1 - \tau)(1 - \phi)\left(-\frac{\beta\alpha}{\phi^2} - \sigma^2(1 - \tau)\right) - (1 - \tau)\left(\frac{\beta\alpha}{\phi} - \phi\sigma^2(1 - \tau)\right) .$$

Plugging these into (IFT), and using (19) and (20), we get after some algebra that:

$$\frac{d\phi}{d\tau} = \phi \frac{(r - R)(1 - 2\phi) - R}{(1 - \tau)[(r - R)(1 - 2\phi) + r]} . \quad (A1)$$

From (19') we have that:

$$\frac{d\mathbf{r}}{d\tau} = -\frac{\alpha\beta}{\phi^2} \frac{d\phi}{d\tau}.$$

In addition, because  $r = \alpha K^{\alpha-1}$ , we also have that:

$$\frac{d\mathbf{r}}{d\tau} = \alpha(\alpha - 1)K^{\alpha-2} \frac{d\mathbf{K}}{d\tau}.$$

Hence, the monotonicity of  $\mathbf{r}(\tau)$  with respect to the tax is the inverse of the monotonicity of  $\phi(\tau)$  with respect to the tax, while the monotonicity of  $\mathbf{K}(\tau)$  with respect to the tax is the same as the monotonicity of  $\phi(\tau)$  with respect to the tax. All three functions,  $\phi(\tau)$ ,  $\mathbf{r}(\tau)$ , and  $\mathbf{K}(\tau)$  attain their respective maximum/minimum at the same point. Also, from (20') we get that:

$$\frac{d\mathbf{R}}{d\tau} = \frac{d\mathbf{r}}{d\tau} - \frac{d\phi}{d\tau} \sigma^2 (1 - \tau) + \phi \sigma^2,$$

and therefore, at the point where the aggregates are maximized,  $dR/d\tau = \phi \sigma^2 > 0$ , i.e. the pre-tax interest rate is increasing in the tax.

Differentiating (17) with respect to the tax and using  $dK/d\tau$ , we get that:

$$\frac{dH}{d\tau} = \frac{d\phi}{d\tau} (\alpha - \phi) \frac{K}{(1 - \alpha)\phi^2} \quad (A2)$$

(ii). Let the after-tax interest rate be  $p \equiv (1 - \tau)R(\tau)$ . Then:

$$\frac{dp}{d\tau} = -R + (1 - \tau) \frac{d\mathbf{R}}{d\tau}.$$

Using (A9), (A8), (A7), (A5), and doing some algebra, we get that:

$$\frac{dp}{d\tau} = \frac{2\phi(r - R)(r + R)}{(r - R)(1 - 2\phi) + r} > 0,$$

because we have shown that the denominator is always positive. Hence, the after-tax interest rate is always increasing with the capital tax. It then follows immediately that the pre-tax interest rate is also always increasing in the capital tax, i.e. that  $dR/d\tau > 0$  always.

(iii) Note that the denominator of (A1) is always positive because  $(r - R)(1 - 2\phi) + r = 2(1 - \phi)(r - R) + R > 0$ , as  $\phi < 1$  and  $r > R$ . Hence, the sign of  $d\phi/d\tau$  depends on the sign of the numerator in (A1). Therefore, we first check whether the numerator can be zero, i.e. whether it can be that  $(r - R)(1 - 2\phi) - R = 0$ . Multiplying both sides with  $(1 - \tau)$ , adding  $2\beta$  to both sides, and using  $\rho = \phi(1 - \tau)(r - R) + (1 - \tau)R$ , we can write the numerator as:

$$(1 - \tau)r - \beta + 2(\beta - \rho) = \beta.$$

But in steady state,  $\rho = m = \beta$ , as can be seen from (15) and the fact that we have logarithmic preferences. Hence, we get that:

$$\frac{d\phi}{d\tau} = 0 \Leftrightarrow (1 - \tau)r = 2\beta.$$

In other words, function  $\phi$  is at a maximum with respect to the capital tax when  $(1-\tau)r = 2\beta$ , or, since  $r \equiv f'(K)$ , when  $K = K^* \equiv (f')^{-1}(2\beta(1-\tau)^{-1})$ . For  $K < K^*$ , we have that  $d\phi/d\tau > 0$ , and for  $K > K^*$ , we have that  $d\phi/d\tau < 0$ . This proves that the function  $\phi(\tau)$  is inversely-U shaped with respect to  $\tau$ , attaining a maximum at the level of the capital stock defined by  $(1-\tau)r = 2\beta$ . Incidentally, this also shows that capital attains a maximum at a level below its corresponding complete-markets steady state level, which is characterized by  $(1-\tau)r = \beta$ . The fact that the steady-state capital stock is below its corresponding complete-markets level also immediately follows by invoking Angeletos (2007), because here  $\theta > (\phi/(2-\phi))$  always, as  $\theta = 1$  here.

**Proof of Proposition 4. Planner's maximization problem.** Plugging (14) into (9), invoking the law of iterated expectations, using the fact that the idiosyncratic shocks cancel out in the aggregate, and using  $x_0 = a_0 + h_0$ , gives equation (27). Equation (34) is factor market clearing. Equation (35) is the government budget constraint. The left-hand-side of equation (36) is the derivative of (27) with respect to the tax at a given point in time. The rest of the planner's constraints have already been discussed in the proofs above and in the main text.

**Mathematical preliminaries for proofs of Lemmas 3 and 4.** Denote the Dirac delta function over the real numbers,  $x \in \mathbb{R}$ , as:

$$\delta(x) = \begin{pmatrix} +\infty & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{pmatrix}$$

where  $\int_{-\infty}^{+\infty} g(x)\delta(dx) = g(0)$  for all continuous compactly supported functions  $g$ . Its cumulative distribution function is the unit step function  $\Delta$ :

$$\Delta(x) = \begin{pmatrix} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{pmatrix}$$

Thus, in particular, the integral of the delta function against a continuous function is:

$$\int_{-\infty}^{+\infty} g(x)\delta(dx) = \int_{-\infty}^{+\infty} g(x)d\Delta(x)$$

We will employ a form of a functional derivative that differentiates with respect to a continuous function, using the Dirac delta as the relevant test function. Given a functional  $F$  and a function  $g$ , the functional derivative of  $F$ , denoted  $\delta F/\delta g$ , at point  $x_0$ , is defined as:

$$\frac{\delta F(g(x))}{\delta g(x_0)} = \lim_{\epsilon \rightarrow 0} \frac{F(g(x) + \epsilon\delta(x - x_0)) - F(g(x))}{\epsilon}$$

The functional derivative describes how the functional  $F(g(x))$  changes as a result of a small change in the entire function  $g(x)$ . The particular form of the change in  $g(x)$  is not specified, but it should stretch over the whole interval on which  $x$  is defined. Employing the particular form of the perturbation given by the delta function has the meaning that  $g(x)$  is varied only in the point  $x_0$ . Except for that point, there is no variation in  $g(x)$ .

**Proof of Lemma 3. Derivatives of  $\tilde{W}$ .** We will be using an infinitesimal Dirac delta function  $\delta_u(s)$ , to indicate a unit impulse at time  $u$ . At that point in time, the capital tax changes unexpectedly, all else equal. Let the second interval in (27) be:

$$F \equiv \int_0^{+\infty} e^{-\beta t} \int_0^t (1-\tau(s))\phi(s)r(s) + (1-\tau(s))R(s)(1-\phi(s)) - \beta - \frac{1}{2}\sigma^2(1-\tau(s))^2\phi^2(s) ds dt$$

Let's first take the derivative of  $F$  with respect to the capital return,  $r$ , at time  $u$ . This return will change at time  $u$ , in response to the change in the tax at that point in time. Using the definition of a functional derivative with the Dirac delta as the test function, we get that:

$$\frac{\delta F(r(s))}{\delta r(u)} =$$

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left\{ \int_0^{+\infty} e^{-\beta t} \int_0^t (1-\tau(s))\phi(s)(r(s)+\epsilon\delta_u(s)) + (1-\tau(s))R(s)(1-\phi(s)) - \beta - \frac{1}{2}\sigma^2(1-\tau(s))^2\phi^2(s) ds dt \right. \\ \left. - \int_0^{+\infty} e^{-\beta t} \int_0^t (1-\tau(s))\phi(s)r(s) + (1-\tau(s))R(s)(1-\phi(s)) - \beta - \frac{1}{2}\sigma^2(1-\tau(s))^2\phi^2(s) ds dt \right\}$$

Simple algebra yields:

$$\frac{\delta F(r(s))}{\delta r(u)} = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_0^{+\infty} e^{-\beta t} \int_0^t (1-\tau(s))\phi(s)\epsilon\delta_u(s) ds dt$$

Simplifying, we get:

$$\frac{\delta F(r(s))}{\delta r(u)} = (1-\tau(u))\phi(u) \int_0^{+\infty} e^{-\beta t} \int_0^t \delta_u(s) ds dt$$

Using the cumulative function of the Dirac delta, we then get:

$$\frac{\delta F(r(s))}{\delta r(u)} = (1-\tau(u))\phi(u) \int_u^{+\infty} e^{-\beta t} dt$$

Evaluating the integral gives:

$$\frac{\delta F(r(s))}{\delta r(u)} = (1-\tau(u))\phi(u) \frac{e^{-\beta u}}{\beta}$$

Following similar steps, we get that:

$$\frac{\delta F(R(s))}{\delta R(u)} = (1-\tau(u))(1-\phi(u)) \frac{e^{-\beta u}}{\beta}$$

Following similar steps, we also get that:

$$\frac{\delta F(\phi(s))}{\delta \phi(u)} = [(1-\tau(u))r(u) - (1-\tau(u))R(u) - \sigma^2(1-\tau(u))^2\phi(u)] \frac{e^{-\beta u}}{\beta}$$

from which, using (10), it follows that:

$$\frac{\delta F(\phi(s))}{\delta \phi(u)} = 0$$

Following similar steps, we also get that:

$$\frac{\delta F(\tau(s))}{\delta \tau(u)} = [-r(u)\phi(u) - (1 - \phi(u))R(u) + \sigma^2(1 - \tau(u))\phi(u)^2] \frac{e^{-\beta u}}{\beta}$$

from which, using (10), it follows that:

$$\frac{\delta F(\tau(s))}{\delta \tau(u)} = -R(u) \frac{e^{-\beta u}}{\beta}$$

**Proof of Lemma 4. Derivative of  $h_0$ .** Note that:

$$h_0 = \int_0^{+\infty} e^{-\int_0^t p(\tau(s))ds} q(\tau(t)) dt$$

where  $p_t = (1 - \tau_t)R_t$  and  $q_t = w_t + T_t = [(1 - \alpha)\alpha^{\alpha/(1-\alpha)} + \tau_t\alpha^{1/(1-\alpha)}]r(\tau_t)^{\alpha/(\alpha-1)}$ , since  $w_t = (1 - \alpha)(r_t/\alpha)^{\alpha/(\alpha-1)}$  and  $T_t = \tau_t r_t^{\alpha/(\alpha-1)}\alpha^{1/(1-\alpha)}$ . For  $\epsilon \rightarrow 0$ , we have that:

$$p(\tau(s) + \epsilon\delta_u(s)) \rightarrow p(\tau(s)) + \epsilon\delta_u(s) \frac{dp}{d\tau(s)}$$

and

$$q(\tau(s) + \epsilon\delta_u(s)) \rightarrow q(\tau(s)) + \epsilon\delta_u(s) \frac{dq}{d\tau(s)}$$

Therefore, using similar steps as above, we get:

$$\frac{\delta h_0}{\delta \tau(u)} = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left\{ \int_0^{+\infty} e^{-\int_0^t [p(\tau(s)) + \epsilon\delta_u(s) \frac{dp}{d\tau(s)}] ds} [q(\tau(t)) + \epsilon\delta_u(t) \frac{dq}{d\tau(t)}] dt - \int_0^{+\infty} e^{-\int_0^t p(\tau(s)) ds} q(\tau(t)) dt \right\}$$

After some algebra and defining  $P(t) \equiv \int_0^t p(\tau(s)) ds$ , we get that:

$$\frac{\delta h_0}{\delta \tau(u)} = e^{-P(u)} \frac{dq}{d\tau(u)} - \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_0^{+\infty} e^{-P(t)} q(\tau(t)) \left( 1 - \int_0^{+\infty} e^{-\int_0^t \delta_u(s) \frac{dp}{d\tau(s)} ds} dt \right)$$

Using the fact that  $e^x = 1 + x/1! + x^2/2! + x^3/3! + \dots$ , we get that the second term of this equation is equal to:

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_0^{+\infty} e^{-P(t)} q(\tau(t)) \left\{ \epsilon \int_0^t \delta_u(s) \frac{dp}{d\tau(s)} ds - \frac{\epsilon^2}{2!} \left( \int_0^t \delta_u(s) \frac{dp}{d\tau(s)} ds \right)^2 + \dots \right\} dt =$$

$$\int_0^{+\infty} e^{-P(t)} q(\tau(t)) \int_0^t \delta_u(s) \frac{dp}{d\tau(s)} ds dt =$$

$$\begin{aligned} \frac{dp}{d\tau(u)} \int_0^{+\infty} e^{-P(t)} q(\tau(t)) \int_0^t \delta_u(s) ds dt = \\ \frac{dp}{d\tau(u)} \int_u^{+\infty} e^{-P(t)} q(\tau(t)) dt \end{aligned}$$

Therefore:

$$\frac{\delta h_0}{\delta \tau(u)} = e^{-\int_0^u p(\tau(s)) ds} \frac{dq}{d\tau(u)} - \frac{dp}{d\tau(u)} \int_u^{+\infty} e^{-\int_0^t p(\tau(s)) ds} q(\tau(t)) dt$$

**Proof of Proposition 5. Optimal tax. (i)** Plug everything from Lemmas 2, 3, and 4 into the planner's first order condition and evaluate as  $u \rightarrow +\infty$ :

$$\begin{aligned} \lim_{u \rightarrow +\infty} \left\{ \frac{1}{\beta} \frac{1}{x_0} (e^{-\int_0^u p(\tau(s)) ds} \frac{dq}{d\tau(u)} - \frac{dp}{d\tau(u)} \int_u^{+\infty} e^{-\int_0^t p(\tau(s)) ds} q(\tau(t)) dt) + \right. \\ \left. (1 - \tau(u)) \phi(u) \frac{e^{-\beta u}}{\beta} \frac{dr}{d\tau(u)} + (1 - \tau(u))(1 - \phi(u)) \frac{e^{-\beta u}}{\beta} \frac{dR}{d\tau(u)} - R(u) \frac{e^{-\beta u}}{\beta} \right\} = 0 \end{aligned}$$

Evaluating the integral in the first term gives:

$$\begin{aligned} \lim_{u \rightarrow +\infty} \left\{ \frac{1}{\beta} \frac{1}{x_0} (e^{-pu} \frac{dq}{d\tau(u)} - \frac{dp}{d\tau(u)} q \int_u^{+\infty} e^{-pt} dt) + \right. \\ \left. \frac{e^{-\beta u}}{\beta} ((1 - \tau(u)) \phi(u) \frac{dr}{d\tau(u)} + (1 - \tau(u))(1 - \phi(u)) \frac{dR}{d\tau(u)} - R(u)) \right\} = 0 \end{aligned}$$

Simple algebra yields:

$$\begin{aligned} \lim_{u \rightarrow +\infty} \left\{ \frac{1}{\beta} \frac{1}{x_0} \left( \frac{dq}{d\tau(u)} - \frac{dp}{d\tau(u)} \frac{q}{p} \right) e^{-pu} + \right. \\ \left. \frac{e^{-\beta u}}{\beta} ((1 - \tau(u)) \phi(u) \frac{dr}{d\tau(u)} + (1 - \tau(u))(1 - \phi(u)) \frac{dR}{d\tau(u)} - R(u)) \right\} = 0 \end{aligned}$$

Rearranging terms yields:

$$\lim_{u \rightarrow +\infty} \left\{ \left( \frac{dq}{d\tau(u)} - \frac{dp}{d\tau(u)} \frac{q}{p} \right) + x_0 \frac{e^{-\beta u}}{e^{-pu}} ((1 - \tau(u)) \phi(u) \frac{dr}{d\tau(u)} + (1 - \tau(u))(1 - \phi(u)) \frac{dR}{d\tau(u)} - R(u)) \right\} = 0$$

Passing the limit inside gives:

$$\left( \frac{dq}{d\tau} - \frac{dp}{d\tau} \frac{q}{p} \right) + x_0 ((1 - \tau) \phi \frac{dr}{d\tau} + (1 - \tau)(1 - \phi) \frac{dR}{d\tau} - R) \lim_{u \rightarrow +\infty} \frac{e^{-\beta u}}{e^{-pu}} = 0$$

From this, because  $p \equiv (1 - \tau)R < \beta$  in steady state, we get that:

$$\frac{dq}{d\tau} - \frac{dp}{d\tau} \frac{q}{p} = 0$$

(ii) Since  $H = q/p$  in steady state, it follows that the above equation is equivalent to:

$$\frac{dH}{d\tau} = 0$$

where  $dH/d\tau$  is given by (A2).

(iii) In steady state, we get from (17) that:

$$\frac{1 - \phi}{\phi} = \frac{H}{K} = \frac{f(K) - (1 - \tau)f'(K)K}{(1 - \tau)RK} > \frac{f(K) - (1 - \tau)f'(K)K}{(1 - \tau)f'(K)K},$$

where the last inequality stems from the fact that  $f'(K) > R$ . Using  $f(K) = K^\alpha$  and  $f'(K) = \alpha K^{\alpha-1}$ , then yields that  $\phi < (1 - \tau)\alpha$ , with equality only when markets are complete. In the first best (complete markets with lump-sum transfers), the optimal tax, evaluated at steady state, is zero, so that  $\phi = \alpha$ .

**Proof of Lemma 5. Risk thresholds.** (i) Combining (18) and (19), we get that:

$$R(1 - \phi)(1 - \tau) = \beta[1 - \alpha(1 - \tau)].$$

Since the left-hand-side of this equation is non-negative, it also has to be that  $1 - (1 - \tau)\alpha \geq 0$ , or equivalently, that  $\tau \geq 1 - 1/\alpha$ . Hence, the minimum feasible tax is  $\tau = 1 - 1/\alpha$ . If capital is inversely-U-shaped with the tax, then the optimal tax cannot be at this minimum. Put differently, the optimal tax can take on its minimum allowable value only when the capital is monotonically falling with the tax, which in turn is only possible in case (iii) – (a) of Proposition 5, where  $\phi = \alpha$ . Therefore, we need to find the threshold  $\underline{\sigma}$ , for which the optimum is characterized by  $\tau = 1 - 1/\alpha$  and  $\phi = \alpha$ . For these values of  $\tau$  and  $\phi$ , condition (19) yields  $r = \beta$ , condition (18) yields  $R = 0$ , and condition (20) yields that  $\sigma^2 = \beta$ . Hence,  $\underline{\sigma} = \sqrt{\beta}$ .

(ii) The threshold  $\bar{\sigma}$  is defined as the level of risk for which, at the optimal steady state, the capital tax is zero and the steady state capital stock is at its maximum. Then, condition (19) yields  $r = \alpha\beta/\phi$ , condition (18) yields  $R = \beta(1 - \alpha)/(1 - \phi)$ , and condition (20) yields that  $R = r - \phi\sigma^2$ . Using the latter two equations, we get that  $\phi^2\sigma^2(1 - \phi) = \beta(\alpha - \phi)$ . From (21),  $\phi$ , and hence  $K$ , are at a maximum if and only if  $(r - R)(1 - 2\phi) - R = 0$ . Plugging in from the steady state conditions, we get that, at  $\tau = 0$ ,  $\phi^2\sigma^2(1 - \phi) = \alpha\beta/2$ . Jointly, these equations imply that  $\phi = \alpha/2$ , and hence that  $\sigma^2 = \frac{4\beta}{\alpha(2-\alpha)}$ , or that  $\bar{\sigma} = \sqrt{\frac{4\beta}{\alpha(2-\alpha)}}$ .

**Proof Proposition 6. Transition.** Let the general equilibrium effect of the tax on asset returns from (47) be  $GE \equiv (1 - \tau)\phi r_\tau + (1 - \tau)(1 - \phi)R_\tau$ . Using (11), (19), (23) and (21), we get that:

$$GE = \frac{r(r - R)\phi + rR + (1 - \phi)(r - R)R}{(r - R)(1 - 2\phi + r)}$$

Using this we then get that:

$$GE - PE = \frac{r(r - R)\phi + rR + (1 - \phi)(r - R)R}{(r - R)(1 - 2\phi + r)} - R$$

which, after some simple algebra, yields:

$$GE - PE = \frac{(r - R)(r + R)\phi}{(r - R)(1 - 2\phi + r)} > 0$$

because we have shown in the proof of Lemma 2 that the denominator is always positive.

Figure 1: Case of First Best  $\sigma = 0$

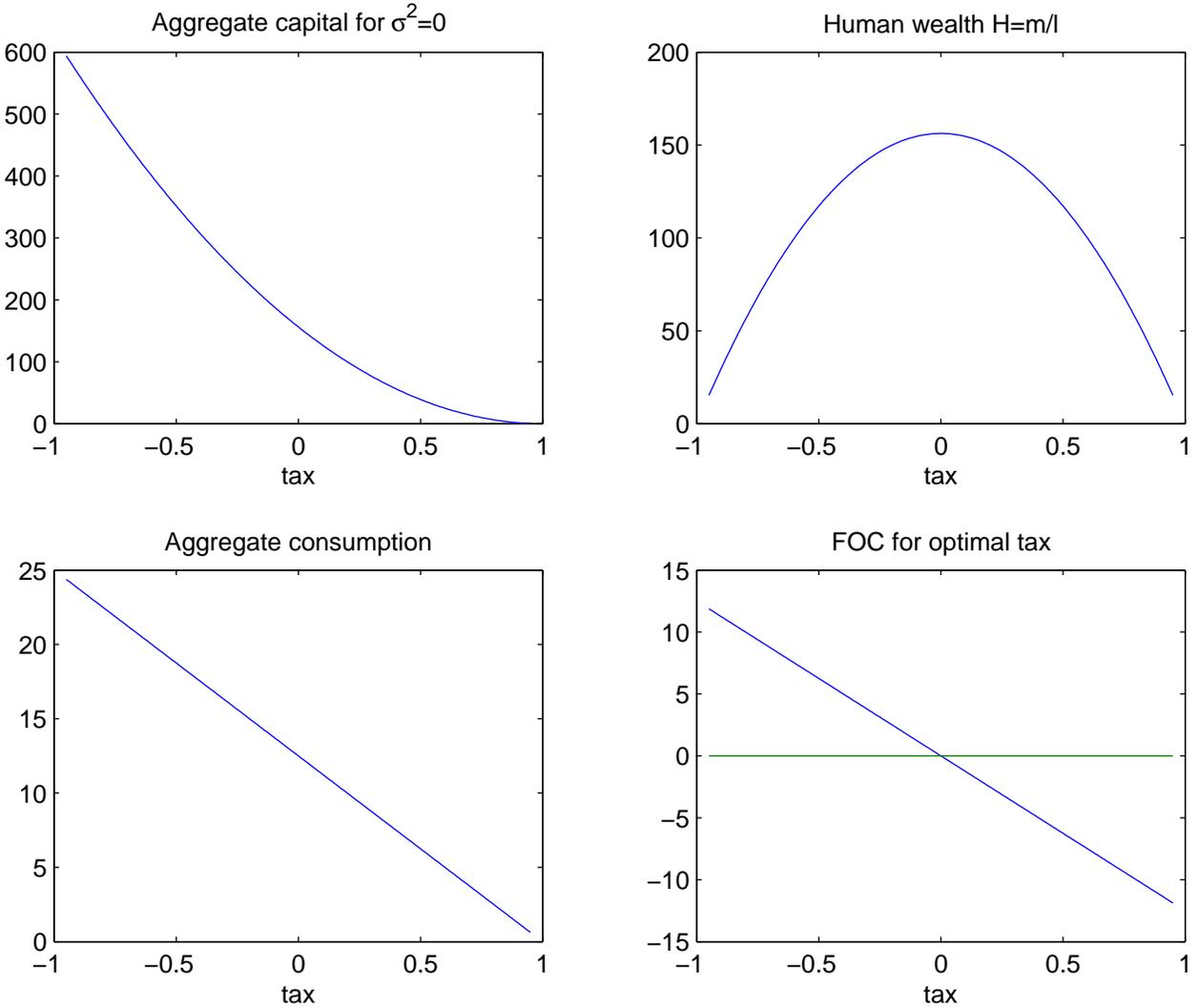


Figure uses parameter values  $\alpha = 0.5$ ,  $\beta = 0.04$ ,  $\sigma^2 = 0$ . The top left panel plots aggregate capital against the capital tax. In complete markets, capital is always decreasing with the tax. The bottom left panel plots aggregate consumption against the capital tax. The top right panel plots human wealth,  $H$ , against the tax. The bottom right panel plots the first order condition (36) against the capital tax. The line cuts the horizontal axis at the ex-ante optimal tax, which is equal to zero. This replicates the complete markets results that the optimal tax is zero. It maximizes human wealth, but not capital or aggregate consumption.

Figure 2: Case of  $0 < \sigma < \underline{\sigma} \equiv \sqrt{\beta}$

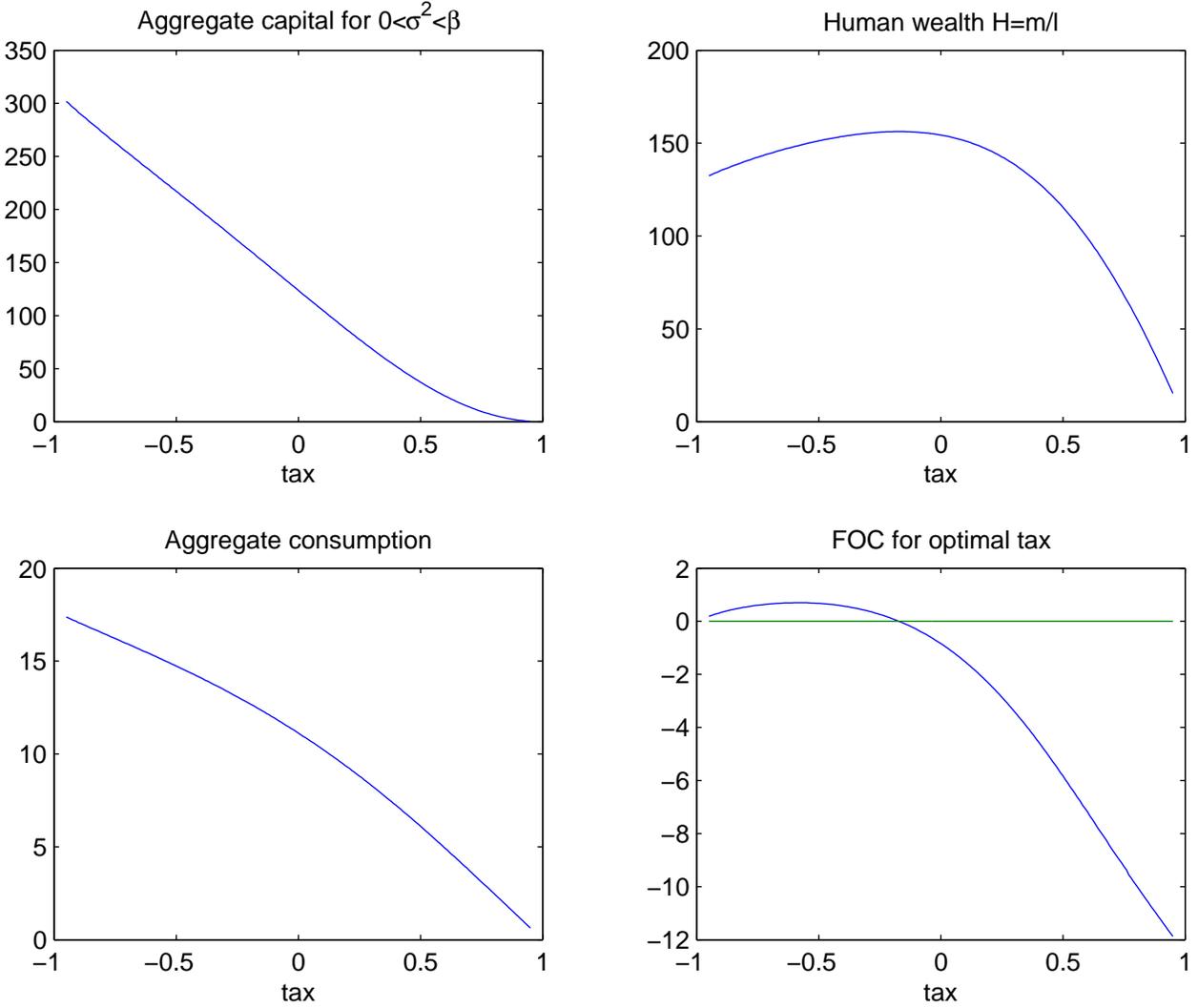


Figure uses parameter values  $\alpha = 0.5$ ,  $\beta = 0.04$ ,  $\sigma^2 = 0.02$ . The top left panel plots aggregate capital against the capital tax. As in complete markets, capital is always decreasing with the tax. The bottom left panel plots aggregate consumption against the capital tax. The top right panel plots human wealth,  $H$ , against the tax. The bottom right panel plots the first order condition (36) against the capital tax. The line cuts the horizontal axis at the ex-ante optimal tax, which is equal to  $\tau = -0.17$ . This is also the value of the tax that maximizes human wealth, but not capital or aggregate consumption.

Figure 3: Case of  $\sigma = \underline{\sigma} \equiv \sqrt{\beta}$

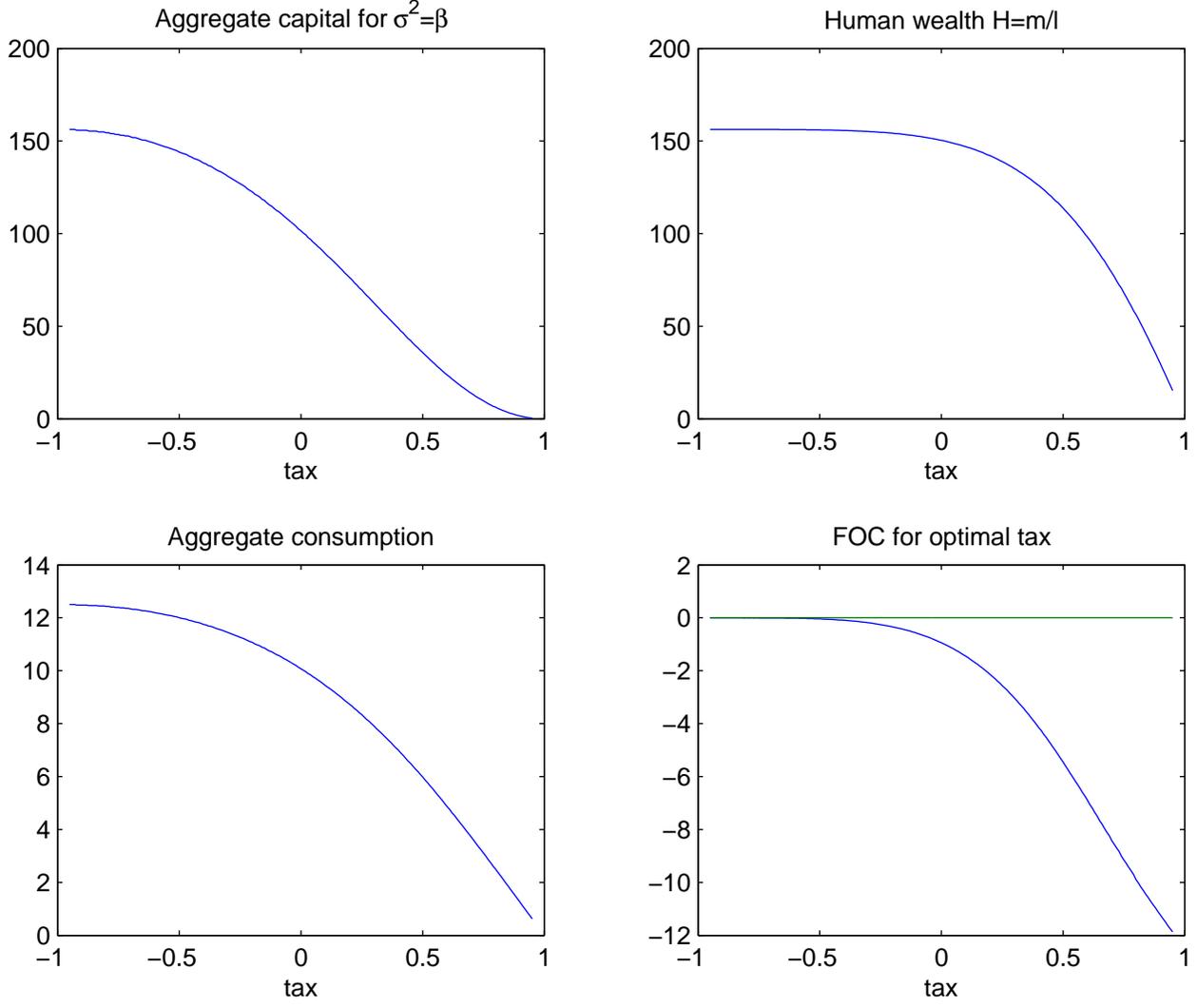


Figure uses parameter values  $\alpha = 0.5$ ,  $\beta = 0.04$ ,  $\sigma^2 = 0.04$ . The top left panel plots aggregate capital against the capital tax. The bottom left panel plots aggregate consumption against the capital tax. The top right panel plots human wealth,  $H$ , against the tax. The bottom right panel plots the first order condition (36) against the capital tax. The line cuts the horizontal axis at the ex-ante optimal tax, which is equal to  $\tau = -1$ , which is the lowest possible tax. This is also the value of the tax that maximizes human wealth, as well as capital and aggregate consumption. This is a borderline case, where capital is always decreasing with the tax, but, at the optimal tax, capital it is at its maximum value.

Figure 4: Case of  $\beta < \sigma < \bar{\sigma} \equiv \sqrt{4\beta/\alpha(2-\alpha)}$

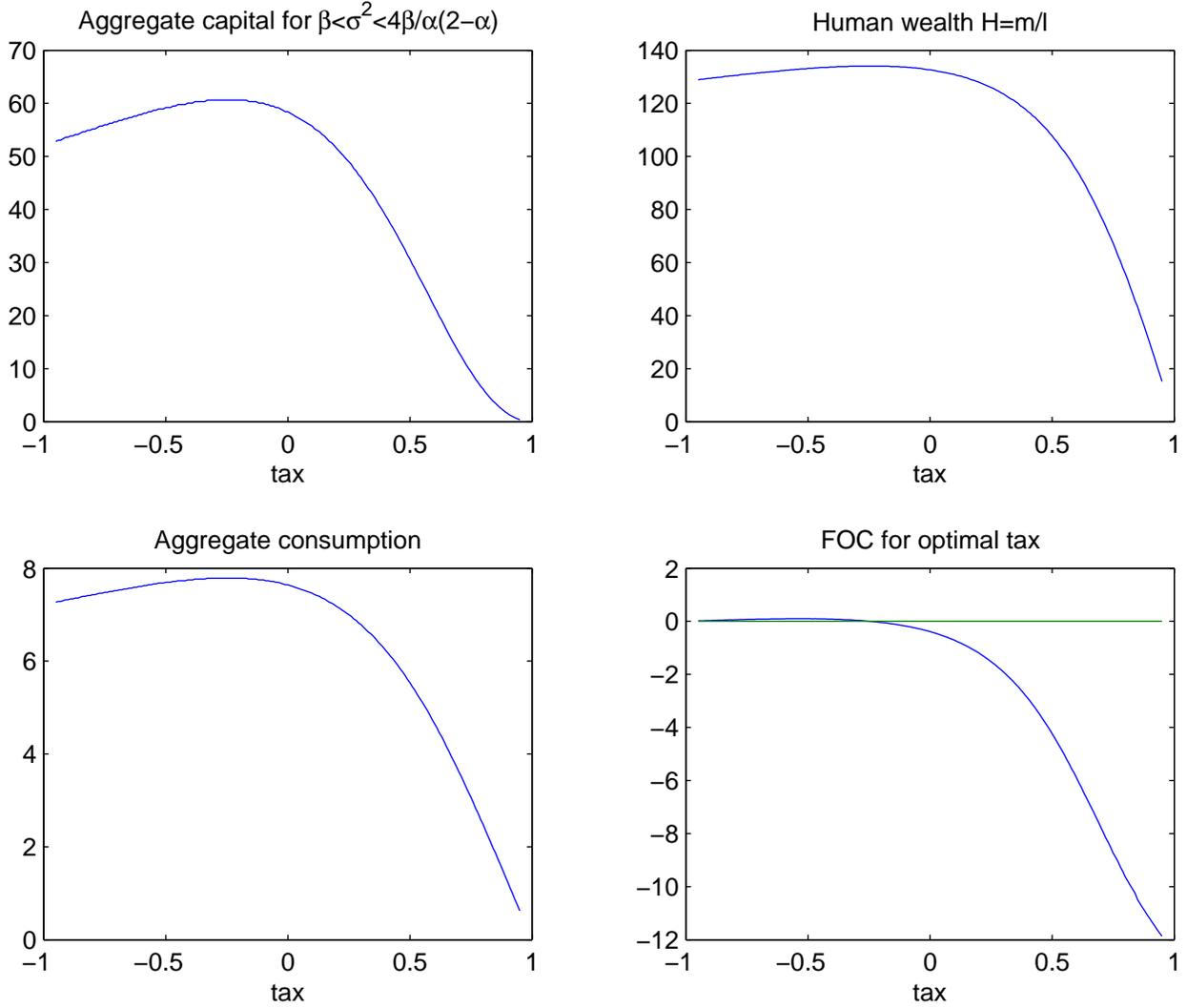


Figure uses parameter values  $\alpha = 0.5$ ,  $\beta = 0.04$ ,  $\sigma^2 = 0.12$ . The top left panel plots aggregate capital against the capital tax, which is a non-monotonic relationship. The bottom left panel plots aggregate consumption against the capital tax. The top right panel plots human wealth,  $H$ , against the tax. The bottom right panel plots the first order condition (36) against the capital tax. The line cuts the horizontal axis at the ex-ante optimal tax, which is equal to  $\tau = -0.25$ . This is also the value of the tax that maximizes human wealth, as well as capital and aggregate consumption.

Figure 5: Case of  $\sigma = \bar{\sigma} \equiv \sqrt{4\beta/\alpha(2-\alpha)}$

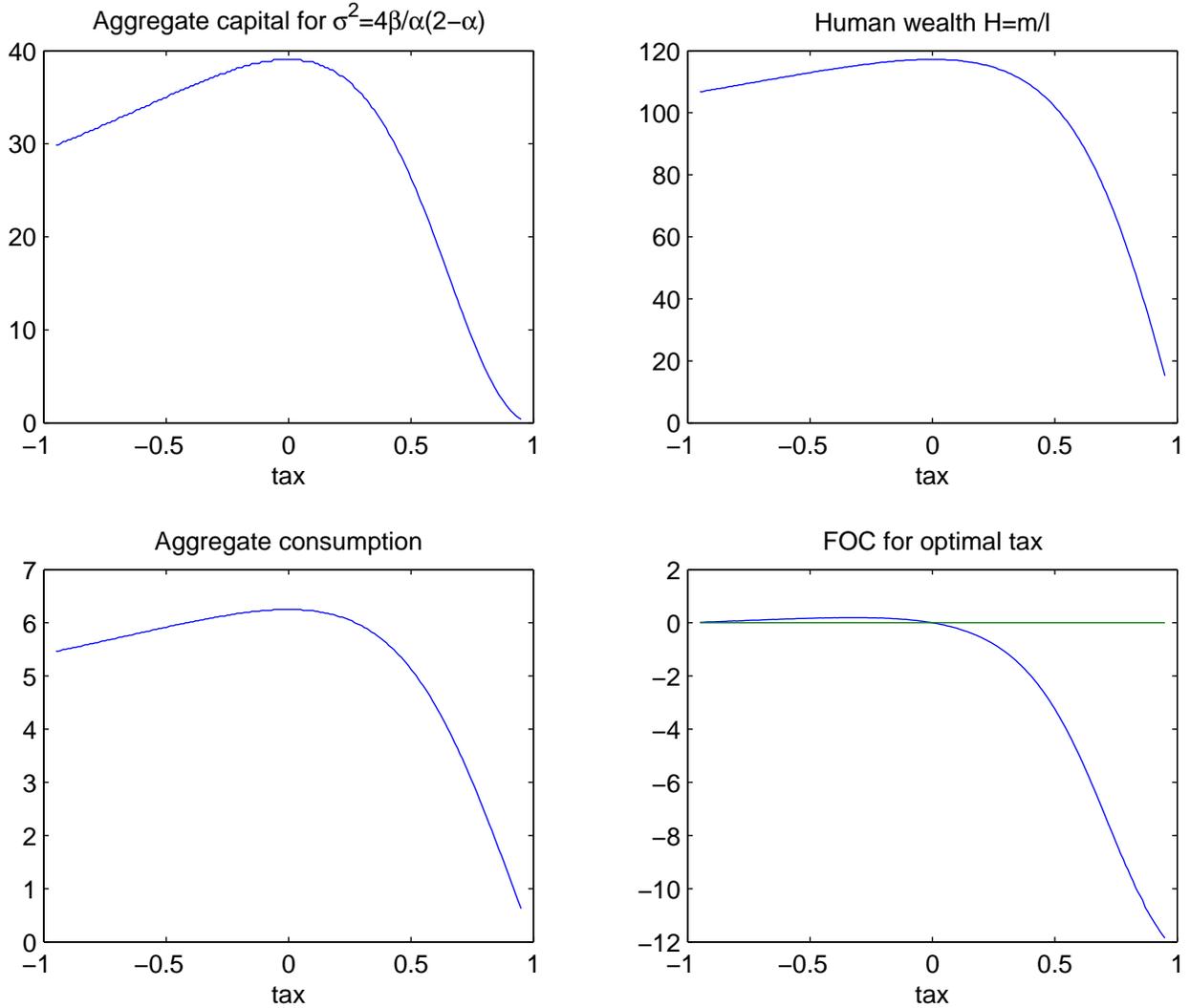


Figure uses parameter values  $\alpha = 0.5$ ,  $\beta = 0.04$ ,  $\sigma^2 = 0.21$ . The top left panel plots aggregate capital against the capital tax, which is a non-monotonic relationship. The bottom left panel plots aggregate consumption against the capital tax. The top right panel plots human wealth,  $H$ , against the tax. The bottom right panel plots the first order condition (36) against the capital tax. The line cuts the horizontal axis at the ex-ante optimal tax, which is equal to  $\tau = 0$ . This is also the value of the tax that maximizes human wealth, as well as capital and aggregate consumption. Although the optimal tax is the same as in complete markets, the steady state values for capital and consumption are significantly lower due to the increased risk in the economy.

**Figure 6: Case of  $\sigma > \bar{\sigma} \equiv \sqrt{4\beta/\alpha(2-\alpha)}$**

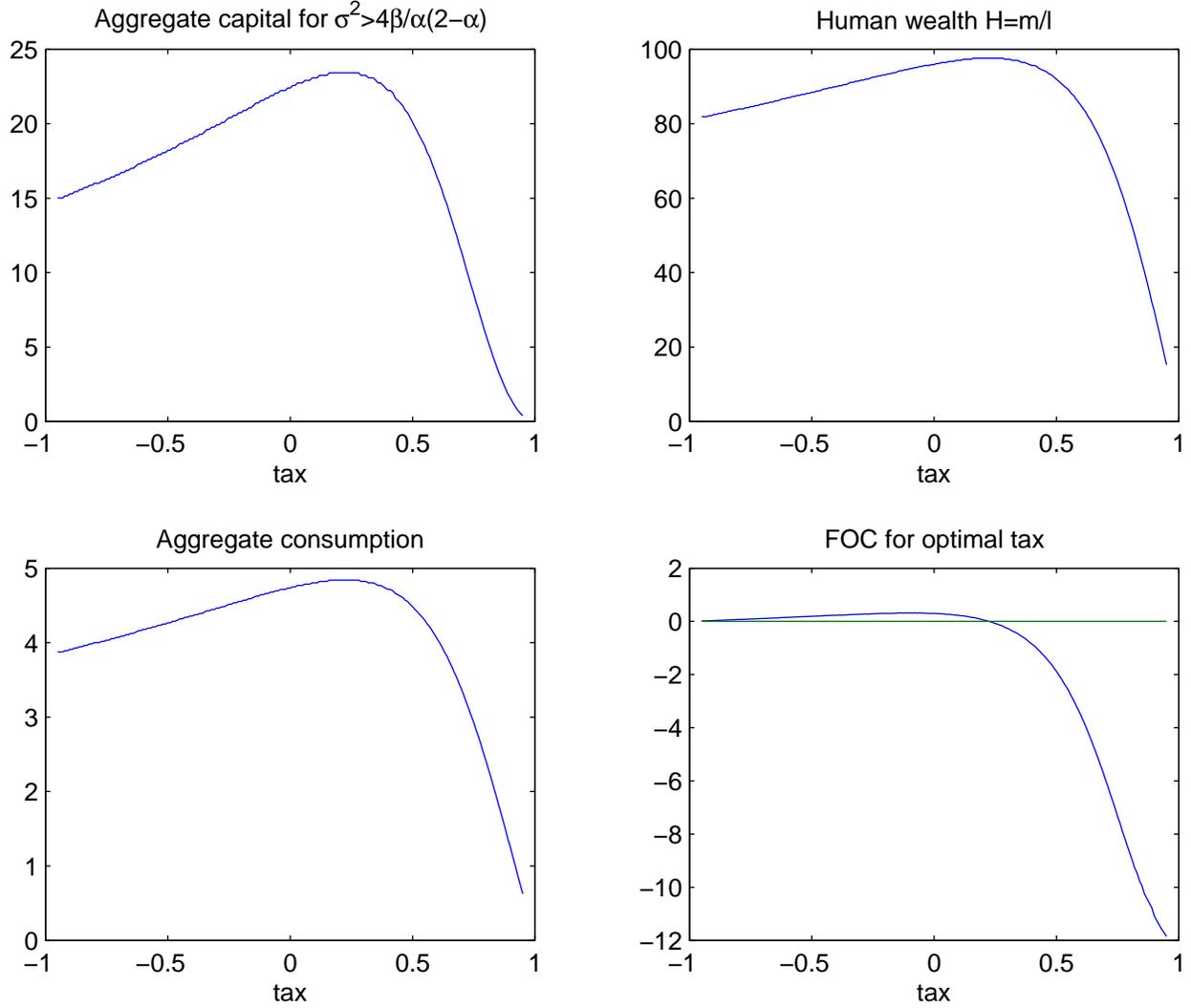


Figure uses parameter values  $\alpha = 0.5$ ,  $\beta = 0.04$ ,  $\sigma^2 = 0.43$ . The top left panel plots aggregate capital against the capital tax, which is a non-monotonic relationship. The bottom left panel plots aggregate consumption against the capital tax. The top right panel plots human wealth,  $H$ , against the tax. The bottom right panel plots the first order condition (36) against the capital tax. The line cuts the horizontal axis at the ex-ante optimal tax, which is equal to  $\tau = 0.22$ . This is also the value of the tax that maximizes human wealth, as well as capital or aggregate consumption. The optimal capital tax will be positive for all values of  $\sigma^2$  higher than  $\frac{4\beta}{\alpha(1-\alpha)}$ .

Figure 7: Optimal tax for varying  $\sigma$

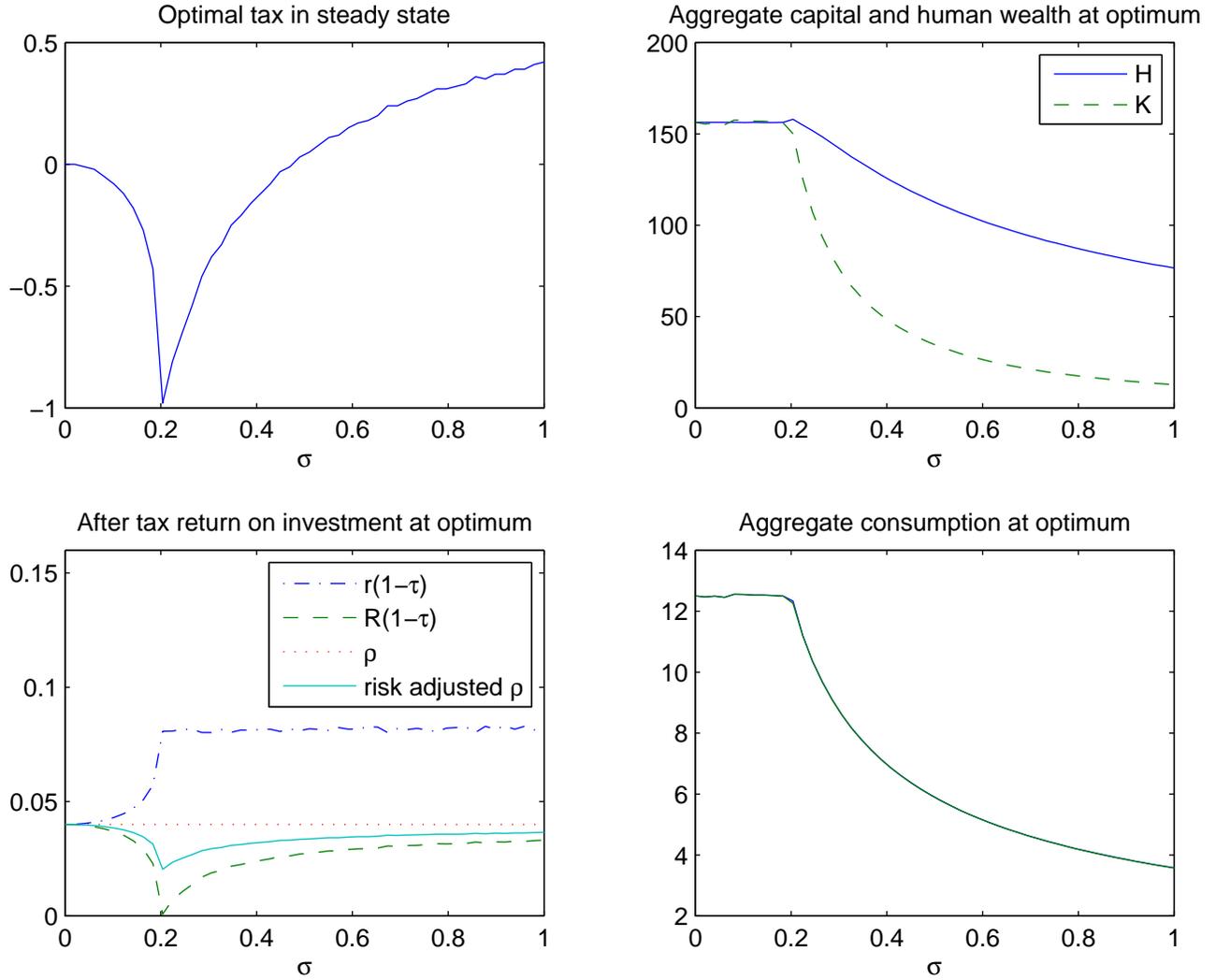


Figure 7 uses parameter values  $\alpha = 0.5$ ,  $\sigma$  between 0 and 2,  $\beta = 0.04$ . The top left panel plots the optimal steady state tax as a function of the varying level of  $\sigma$ . The optimal tax starts at zero and is initially decreasing with the level of uncertainty. For higher values of  $\sigma$  it becomes increasing with uncertainty, and eventually becomes positive. The top right panel plots the level of steady state aggregate capital and human wealth for different values of  $\sigma$  evaluated at the optimal tax. For low values of sigma, where the tax is decreasing with uncertainty, aggregate capital is at its first best level, and is decreasing afterwards. The same happens for human wealth. The bottom right panel shows aggregate consumption, which follows a similar pattern to capital and human wealth. The bottom left panel plots  $\rho$  and the risk adjusted return to saving as well as the after tax return on risky and riskless assets.